Noise Induced Dissipation in Discrete-Time Classical and Quantum Dynamical Systems

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Abstract

In this dissertation, written under the supervision of Prof. Albert Fannjiang, we study statistical and ergodic properties of randomly perturbed (noisy) classical and quantum dynamical systems. We concentrate on the discrete time dynamics generated by Lebesgue measure preserving maps defined on d-dimensional torus. We introduce the notion of the *dissipation time* which enables us to test how the system responds to the noise and in particular to measure the speed at which an initially closed, conservative system converges to the equilibrium when subjected to noisy interactions with its environment.

We study the asymptotics of the dissipation time in the limit of vanishing noises and prove that it provides a robust criterion of the chaoticity of the underlying conservative system. The results formalize in a rigorous and quantitative way the idea that the dissipation is *fast* for chaotic systems and *slow* for regular ones. In the classical setting, we show that chaotic systems, e.g., Anosov diffeomorphisms possess logarithmic dissipation time while for non-chaotic maps the corresponding asymptotics is of a power-law type. In case of diagonalizable ergodic toral automorphisms we compute the exact value of the dissipation rate constant and show that it is equal to the reciprocal of the minimal dimensionally averaged KS entropy among all irreducible components of the rational block diagonal decomposition of the map.

In case of quantum systems we introduce the notion of the dissipation time for both finite and infinite dimensional quantizations on the torus. We study the simultaneous semiclassical and small noise asymptotics of the quantum dissipation time and relate it to the notions of the Ehrenfest time and the dynamical entropy of the quantum system. We concentrate on quantum toral symplectomorphisms (generalized cat maps) for which we compute the exact asymptotics of their quantum dissipation time and show that it coincides with a classical one in the semiclassical regime in which the magnitude of the Planck constant does not exceed the the size of the noise.

Chapter 1

Introduction

The main subject of this dissertation is the study of statistical and ergodic properties of noisy dynamical systems. We investigate the problems of irreversibility and approach to equilibrium for randomly perturbed classical and quantum systems exhibiting various degrees of chaoticity.

The origin of irreversibility in dynamical systems is usually modeled by small stochastic perturbations of the otherwise reversible evolution. These perturbations may be attributed to many different sources: uncontrolled interactions with the environment, internal stochasticity of the system or unavoidable simplifications made in theoretical models of real-life experiments; e.g., some internal variables neglected in the equations. In experimental or numerical investigations, stochasticity or noise is introduced respectively by finite precision of the preparation and measurement apparatus, and by rounding-off errors due to finite precision of numerical computations. The important common feature is that noises, intrinsic (internal stochasticity) or extrinsic (random influence from the environment), can, on appropriately long time scales, induce or emphasize effects that would be absent or difficult to discern without noise.

In this work we concentrate mainly on one such effect, the effect of *dissipation*.

The term dissipation refers in our study to the loss of the energy of fluctuations of densities, or equivalently, observables of the system during the course of noisy evolution. The strength of dissipation can be determined by measuring the speed at which an initially closed, conservative system converges to the equilibrium when subjected to noisy interactions with the environment. The latter task can be accomplished by studying an appropriate time scale on which the influence of the noise becomes noticeable, i.e., affects the dynamics on characteristic spatial scales of the whole system. Such time scale will be called the *dissipation time* and will constitute the main object of our study.

Intuitively speaking, the dissipation time is a time scale on which the magnitude of initial density fluctuations is brought below a certain fixed threshold and hence the system finds itself in an intermediate state, roughly speaking, 'half-way' from its final equilibrium. From a physical point of view, the dissipation time captures the time scale on which the system, due to the action of environmental noise, achieves a certain fixed level of Boltzmann-Gibbs entropy (cf. Section 2.2.3).

The main goal of this work is to determine the relation between *ergodic*, and in particular *chaotic*, properties of unperturbed, conservative systems and *dissipative* properties of their noisy counterparts. The main method is to study the asymptotics of the dissipation time in the limit of vanishing noises. Our main task in the first part of the work is to characterize in a rigorous and quantitative way the rate of the dissipation for classical systems. The results will support and formalize an intuitive understanding that dissipation should be *fast* for chaotic dynamics and *slow* for a regular one, the difference being more and more visible as the magnitude of the noise decreases. The fact that we are mainly interested in the small noise limit has a direct physical interpretation.

Indeed, in the experimental setting considerable effort is usually made to eliminate the influence of the noise on the system by isolating it from its environment

(at least to some reasonable degree). It is, however, impossible to eliminate the noise completely. In the theoretical approach such situations are usually modeled by limiting procedures (the magnitude of the noise is positive but assumed to be arbitrary small).

The notion of the dissipation time, as described above and defined in Section 2.2.2, is relatively new. It has been introduced in the context of classical, continuous-time systems in [48, 49] (cf. Section 2.2.1) and later developed and extended to discrete-time classical and quantum systems in the following series of works [50, 51, 52].

One of the most important findings is that the asymptotics of the dissipation time provides clear and robust characteristics of chaoticity of underlying conservative (noiseless) systems. To explain the connection between the dissipation time and chaoticity we need to review briefly some basic notions from the theory of chaotic systems and relate them to our results.

Chaotic behavior of classical dynamical systems has been studied with an exceptional intensity over the period of the last fifty years. Many equivalent ways of defining or characterizing chaos were developed over that time. Among the most important and well-known approaches to chaoticity one has to mention at least the following

- *Algebraic approach*: K-systems. In this approach chaoticity is characterized through the Kolmogorov property (see Definition 3.1), which encodes in an algebraic language the idea that the system, although deterministic, behaves effectively like a memoryless process. This approach allows for some generalization to quantum systems ,i.e., to the noncommutative algebraic setting with the classical definition recovered as a particular (commutative) case (cf. Section 3.1.1 and Chapter 4).

- *Geometric approach*: Hyperbolicity. Chaoticity is characterized here by uniform hyperbolicity ,i.e., strict positiveness of all Lyapunov exponents. The condition expresses the geometric picture of two nearby trajectories separating from each other at an exponential rate in time. Different, slightly weaker, formulations are also allowed in this approach, e.g., almost uniform hyperbolicity or quasi-hyperbolicity (with a typical example given by ergodic but not hyperbolic toral automorphisms, cf. [12, 108]).

- *Ergodic approach*: Strong mixing. In this approach, fast, i.e., at least exponential mixing is required if the system is to be called chaotic. This characterization is especially useful if the dynamics is modeled on the level of densities or observables of the system (not directly on the phase space). In particular the property is equivalent to fast (at least exponential) decay of correlations and the approach is particularly useful in spectral analysis (cf. Sections 2.3 and 2.6).

- *Entropic approach*: Positiveness of KS Entropy. The system is called chaotic here if it has *completely positive* Kolmogorov-Sinai (KS) entropy. The term completely positive entropy [106] refers to the property that the entropy of *any* partition of the phase space is strictly positive. Mere positiveness of KS entropy do not guarantee chaoticity of the system, as it is not difficult to construct an example of a nonergodic map with positive entropy (an example will be given later in this Introduction). One of the advantages of this approach lies in the fact that it gives a clear information-theoretical interpretation of the notion of chaoticity.

We note that in the classical setting some of these approaches are equivalent. For example, by Pinsker theorem [106] (see also [110]), the system has K-property iff it has completely positive KS entropy. However, it is worth noting here that some attempts to generalize both notions to quantum systems led to nonequivalent

counterparts (for more details see Chapter 4 or [19]). Some of these approaches allow one to introduce different degrees of chaoticity. In the geometric approach, different levels of hyperbolicity can be specified. In the ergodic formulation, one may require a particular speed of decay of correlations within a particular class of observables determined, e.g., by some regularity properties.

In this dissertation we introduce another characteristics of chaoticity, namely *the asymptotics of the dissipation time*. The difference between the above-mentioned approaches and the present one lies in the fact that chaoticity is tested here in an extrinsic way. We test how the system responds to the noise. Given this information, and not necessarily knowing all the details of the underlying conservative dynamics, one can distinguish chaotic from regular behavior. This makes the dissipation time a robust criterion of chaoticity (cf. remark before Proposition 3.9 in Section 3.1.2). Moreover, since all real-life experimental systems are inevitably subject to noisy interactions with their environments the present approach is well suited for practical purposes.

Another reason for considering the asymptotics of the dissipation time as a test of chaoticity is that the notion can be almost literally and quite successfully adapted from the classical to the quantum setting. Similarly as for the classical dynamics, the quantum dissipation time provides here a good criterion which enables us to distinguish chaotic from regular behavior in an appropriate semiclassical regime (cf. Section 5.4.2). The importance of this observation can be understood if one takes into account the fact that despite great progress made in the field of quantum chaos in the last two decades, there is still no agreement on what quantum chaoticity really means. The main problem lies in the fact that one cannot simply take any particular classical definition of chaoticity and apply it directly to a quantum system. Indeed, as mentioned above, in classical dynamics chaoticity is usually connected with the notion of a trajectory of a system and the arbitrary closeness of two nearby trajectories (the

geometric approach), or equivalently with theoretical ability to resolve the details of the phase space to arbitrary small scales (the entropic approach). For obvious reasons neither of these notions has any direct counterpart in quantum case. On the other hand, even if some quantum generalization can be constructed (e.g., via the algebraic approach), the way to obtain it is usually non-unique, and the same classical notion can have many nonequivalent quantum counterparts (for more detailed discussion and references see Chapter 4). The theory of the quantum dissipation time developed in this dissertation can be viewed as one of the many possible ways of approaching the problem of chaoticity in quantum systems. The second part of this work will be entirely devoted to this subject.

Now we pass to a more systematic discussion of our main results. We start with the classical setting considered in the first part of the dissertation. One general comment is appropriate here. Namely, the notion of the *classical* dissipation time is independent of any particular mathematical model of the noisy dynamical system one chooses to work with. However, in order to fix the attention and, more importantly, be able to derive concrete, rigorous results one needs to choose a certain class of models for which a uniform framework can be constructed and the results for different systems can be compared, provided that they belong to the selected class. In this dissertation we choose to work with discrete time systems (maps) defined on compact phase spaces (represented by d-dimensional tori) with Lebesgue measure as a natural invariant measure for both conservative and noisy dynamics.

As a matter of illustration and to build up some intuition we start by presenting some simple but representative examples of classical maps for which exact results regarding the asymptotics of their dissipation time are available. The simplest examples (toy models) of that kind are as follows

I. **Translations on** $\mathbb{T}^d$, defined by $F\boldsymbol{x} = \boldsymbol{x} + \boldsymbol{v}$, represent the simplest examples of nonergodic, if $1, v_1, .., c_d$ are rationally dependent, or otherwise ergodic but

Map	Ergodic Properties	Dissipation Time
I. Translations	Not ergodic or not weakly-mixing	$\tau_c = \epsilon^{-2} + \mathcal{O}(1)$
II. Cat maps	Exponentially mixing	$\tau_c = \frac{2}{h(F)} \ln(\epsilon^{-1}) + \mathcal{O}(1)$
III. Angle doubling	Exponentially mixing	$\tau_c = \frac{1}{\ln 2} \ln(\epsilon^{-1}) + \mathcal{O}(1)$

Table 1.1: Asymptotics of dissipation time for typical maps

not weakly-mixing maps.

II. **Cat maps on** $\mathbb{T}^2$, i.e., elements $F \in SL(2,\mathbb{Z})$ satisfying $|\operatorname{Tr} F| > 2$ projected on the torus (see [10]), provide simple examples of uniformly hyperbolic, exponentially mixing, fully chaotic systems.

III. **Angle doubling map** $Fx = 2x \bmod 1$ provides an example of a uniformly expanding, exponentially mixing, noninvertible chaotic map.

Let ϵ denote the strength of the noise and T_ϵ the the operator representing the action of the noisy dynamics associated with the above conservative maps on the observables of the system. The classical dissipation time τ_c ("c" stands for "classical") is defined as follows (for detailed definition see Section 2.2.2).

$$\tau_c = \min\{n \in \mathbb{Z}_+ : \|T_\epsilon^n\| < e^{-1}\},$$

where $\|T_\epsilon^n\|$ denotes the operator norm of T_ϵ^n.

The corresponding asymptotics of τ_c as a function of positive, but vanishing magnitude of the noise ϵ is summarized in Table 1.1

Two observations emerge from the above picture. The first general observation is that two qualitatively different behaviors of the dissipation time can immediately be noticed:

- If the dynamics is *regular* then $\tau_c(\epsilon^{-1})$ diverges (as ϵ vanishes) in a power-law fashion. Here we speak of *slow* or *simple* dissipation (long dissipation time).

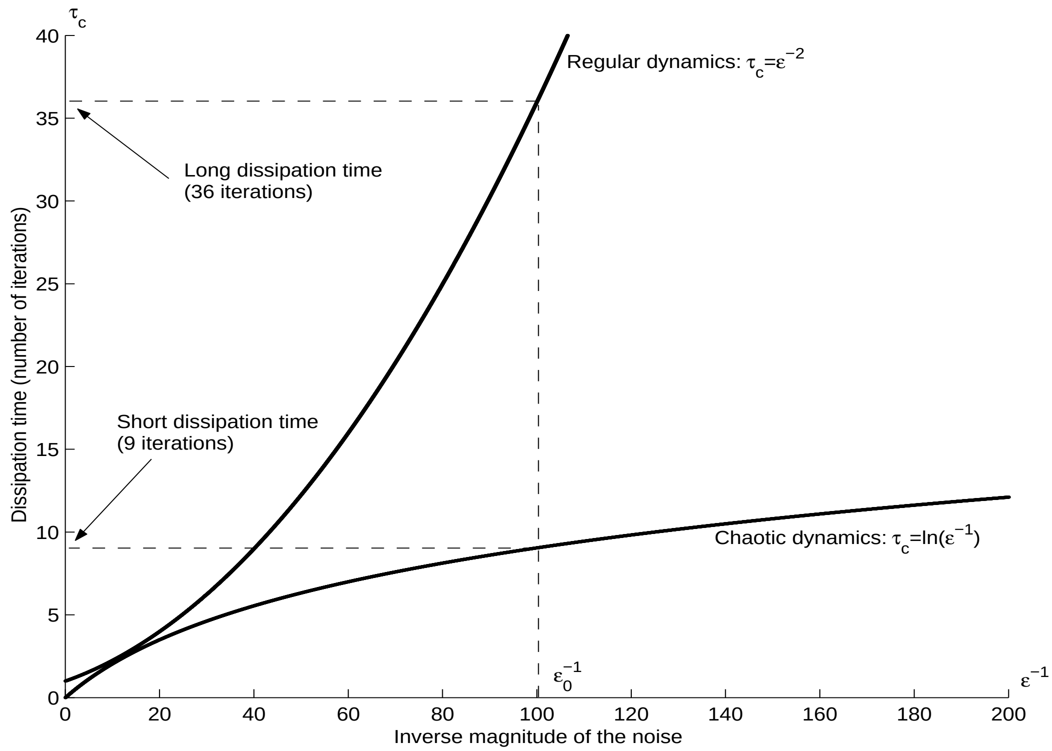

Figure 1.1: Asymptotics of dissipation time

- If the dynamics is chaotic then $\tau_c(\epsilon^{-1})$ has logarithmic asymptotics. In this case we speak of *fast* dissipation (short dissipation time).

We note that when the rate of divergence of τ_c as a function of ϵ^{-1}, with $\epsilon \to 0$, is fast then the dissipation is slow (dissipation time is long), and vice versa.

Figure 1.1 illustrates both behaviors and explains this terminology. Indeed, the number of iterations required to keep $\|T_\epsilon^n\|$ at a constant level (here e^{-1}), i.e., the dissipation time is plotted here against the inverse magnitude of the noise for typical regular (upper curve) and chaotic (lower curve) systems. If we fix some small amount of noise, say $\epsilon_0 = 10^{-2}$, the reduction of the norm of T_ϵ^n to the prescribed level is achieved much faster (9 iterates) in a chaotic (logarithmic) regime than in a regular (power-law) one (36 iterates).

The second observation is of a rather particular nature. Namely, we note that in

the case of these simple chaotic systems, the constant of the logarithmic asymptotics is reciprocally proportional to their Kolmogorov-Sinai entropy $h(F)$. As we will see later this observation does not generalize in any obvious way to higher dimensions (cf. Theorem 3.7) and to more complicated maps.

In view of the above observations a natural question arises whether the power-law and the logarithmic are the only possible asymptotic behaviors of the dissipation time.

To address this question, we first note that it is possible to provide quite sharp general upper and lower bounds for the dissipation time within a vast class of dynamical systems. Indeed, using methods from the spectral theory of non-normal operators and in particular the notion of the pseudospectrum (see Section 2.3), and investigating basic geometric properties of conservative maps (hyperbolicity and local expansion rates studied in Section 2.5), we arrive at the following results :

- The dissipation time of an arbitrary dynamical system generated by a measure-preserving map is never finite (i.e. $\tau_c \to \infty$, as $\epsilon \to 0$).

- The rate of divergence of τ_c is never faster than power-law in ϵ,

$$\tau_c \leq \epsilon^{-\alpha}, \alpha \in (0, 2].$$

- If the map F is C^1 then the divergence of τ_c is never slower than logarithmic:

$$\frac{1}{\ln \|DF\|_\infty} \ln(\epsilon^{-1}) \leq \tau_c,$$

  where $\|DF\|_\infty = \sup_{\boldsymbol{x} \in \mathbb{T}^d} \|(DF)(\boldsymbol{x})\|$ denotes the highest expansion rate of F (if $\|DF\|_\infty = 1$ then $\tau_c \sim \epsilon^{-\alpha}$).

- Almost all non weakly-mixing systems (a modicum of regularity is required, cf. Theorem 2.12) undergo power-law (i.e., the slowest possible) dissipation: $\tau_c \sim \epsilon^{-\alpha}, \alpha \in (0, 2]$.

The question which arose from the first observation is now reduced to the problem of establishing an logarithmic upper bound for the dissipation time of a largest possible class of systems exhibiting some chaotic properties, or equivalently deciding whether there exists a map for which an intermediate (i.e., contained strictly between power-law and logarithmic) asymptotics hold.

The second observation suggests that for systems with logarithmic dissipation time, the value of the dissipation rate constant (i.e., the prefactor of the asymptotics) should provide valuable information about the underlying conservative dynamics.

Chapter 3 is entirely devoted to the study of these two problems.

As to the first problem, we developed two different methods which allowed us to establish logarithmic asymptotics respectively for linear (toral automorphisms - Section 3.1.2) and nonlinear (C^3 Anosov diffeomorphisms - Section 3.3) hyperbolic maps in arbitrary phase space dimension.

Both methods rely eventually on quite advanced number theoretical or respectively spectral analysis and it seems that there is no 'short-cut' way to establish logarithmic asymptotics for any chaotic dynamical system except for simple 1- or 2-dimensional toy models (e.g., cat maps).

As far as abstract (i.e., not related to any particular map) results are concerned, we derive in Section 2.6 a general connection between mixing properties (the rate of decay of correlations) of both conservative and noisy dynamics and the rate of the divergence of the dissipation time. In particular we show that within a large class of maps, strong (exponentially fast) mixing implies logarithmic dissipation time. On the other hand we also prove that methods used in the computation of the dissipation can be used to determine in certain cases the (precise) rate of decay of correlations (cf. Proposition 3.9 in Section 3.1.2).

Let us also comment here briefly on the second problem. The exact solution is now only available in the case of diagonalizable toral automorphisms. The result is

established in Theorem 3.7, where the dissipation rate constant is proved to be equal to the reciprocal of the minimal dimensionally averaged KS entropy among irreducible components of the rational block diagonal decomposition of the map. At this point another question arises: why does the minimal dimensionally averaged KS-entropy appear in the constant instead of KS-entropy itself?

The complete answer to this question is not known. However, the following considerations can shed some light on it. It is known that the knowledge of KS-entropy itself (e.g., its positiveness) is not sufficient to determine whether the system is chaotic or not. Indeed, consider the following toral automorphism in 4-dim represented in the block-diagonal form

$$F = \begin{bmatrix} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix} \end{bmatrix}$$

The first block is simply the identity and the second is a hyperbolic automorphism (the famous Arnold's cat [10]). The entropy of F is positive and equals the entropy of Arnold's cat, but the system is not even ergodic (toral automorphisms are ergodic iff no root of unity lies in their spectrum - see Section 3.1.1). The minimal dimensionally averaged entropy is in this case 0 and the system undergoes slow (power-law) dissipation characteristic for non-chaotic systems.

The fact that the dissipation rate constant averages the KS entropy over the dimension of the irreducible block is of separate importance. We will not explore it fully here. Let us only mention the following simple example. Consider two matrices $F_1 \in SL(d_1, \mathbb{Z})$ and $F_2 \in SL(d_2, \mathbb{Z})$ and assume that they have the same or almost the same spectra, but with different degeneracies. If it happens that $d_1 \gg d_2$ then also $h_{KS}(F_1) \gg h_{KS}(F_2)$ while their dimensionally averaged counterparts are of the same order $\hat{h}(F_1) \sim \hat{h}(F_2)$. This reflects the natural intuition that if the strengths of

Lyapunov exponents of two systems are comparable then the degree of their chaoticity should also be comparable (i.e., independent of the dimension).

As far as nonlinear maps in the context of the second problem are concerned, we derive lower and upper bounds for the dissipation rate constant for C^3 Anosov systems (Theorem 3.24) but the exact value (and even its existence, not to mention its connection to the KS entropy) remains unknown.

To conclude the description of the first part of the dissertation we want to mention that in Section 3.2 we collect some of the many possible generalizations and applications of the above described results, especially the ones concerning toral automorphisms. In particular we investigate the possibility of defining the dissipation time for maps with degenerate noise kernels (Section 3.2.3), and we study the relations between the asymptotics of the dissipation time and some typical time scales encountered in the study of the so-called kinematic dynamo problem (Section 3.2.4).

Now we pass to the description of the second part of the present work, devoted to the study of the dissipation time in the quantum mechanical setting.

The second part begins in Chapter 4, called the *Interludium* as it constitutes a separate and almost independent part of this work. It is meant as a historical overview and in the same time as a quick but comprehensive introduction to the specific area of quantum mechanics on the torus. We describe there the two most important and most commonly encountered in the literature quantization schemes for toral maps. We put a special emphasis on careful explanation of their origins. We also discuss similarities and differences between these two approaches. In particular, we concentrate on the role which semiclassical analysis of spectral properties of quantum chaotic systems on the one hand, and the introduction of several nonequivalent notions of quantum dynamical entropy, on the other, played in the development of both quantization methods (and vice versa).

In the first part of Chapter 5 (cf. Sections 5.1 and 5.2) both quantization meth-

ods are presented in a systematic and rigorous way. We develop a general framework (based on Weyl quantization), which unifies both approaches (i.e., each can be derived as a special case of the general scheme). This prepares the ground for an introduction of the quantum noise (Section 5.3) and the notion of quantum dissipation time (Section 5.4). The final part of the chapter is devoted to our results on semiclassical analysis of the dissipation time of canonical toral maps (Sections 5.4.1 and 5.4.2).

Before we discuss these results in more detail a word of caution is necessary here. Namely, as mentioned above and explained in the Interludium, the quantization of any classical system and in particular a toral canonical map can be performed in many different ways. We would like our results to depend as little as possible on the particular quantization scheme and this is why certain effort was made in this work to present the results in most unified way possible. Nevertheless, complete independence is not possible and it is necessary to describe precisely the quantization principles and methods adopted in a given approach before the results can be stated (this non-uniqueness of the setting and dependence on quantization procedures constitutes one of the most fundamental differences between classical and quantum descriptions). We sketch some of the quantization principles briefly here and refer to Chapters 4 and 5 for a detailed presentation.

Quantization on the torus is usually approached from the following two, non-equivalent points of view:

- *Finite dimensional approach.* This method (sometimes referred to as canonical quantization) was originally introduced in [63], and later generalized and developed in a number of works [43, 45, 70, 24, 108].

- *Infinite dimensional approach.* Usually referred to as *-algebraic noncommutative deformation of the torus, see, e.g., [19, 73].

In this work we do not distinguish between these two approaches in terms of

whether they are 'canonical' or 'algebraic' (cf. [40]). The reason is that the real difference between these two quantizations lies in the choice of the fundamental Hilbert space of pure quantum states of the system whose classical counterpart has the $2d$-dimensional torus as a phase space. After the choice is made (i.e., the space is chosen to be either finite or infinite dimensional), both quantizations can be studied in any, including in particular 'abstract' *-algebraic, framework.

It turns out, however, that the finite dimensional approach is much more suitable for semiclassical analysis of the dissipation time, and most of the results of Chapter 5 are stated in this setting. For completeness and to illustrate the difference we also consider the infinite dimensional case. In particular we prove that in this case, regardless of the value of the Planck constant, quantum and classical evolutions of toral symplectomorphisms coincide.

In the finite dimensional setting, the geometry of the torus (the phase space) and standard requirements of quantum mechanics (conjugacy between the position and momentum representations) restrict the space of admissible wave functions (pure states) to quasiperiodic Dirac delta combs. Planck's constant $h = 2\pi\hbar$ is then restricted to reciprocals of integers $N \in \mathbb{Z}_+$, and the resulting quantum Hilbert space of pure states is N^d-dimensional (for detailed explanations see Section A.2 in Appendix A). The unitary and noisy quantum dynamics can be implemented either on the set of all quantum states, i.e., density operators (the Schrödinger picture corresponding to the Frobenius-Perron approach in classical case) or on the N^{2d}-dimensional algebra of quantum observables (the Heisenberg picture - quantum counterpart of classical the Koopman formalism).

For a fixed finite dimensional quantum system (i.e., fixed Planck constant) and vanishing noise the presence of a pure point, unitary spectrum of the quantum propagator forces the dissipation time to have the same, trivial (i.e., power-law) asymptotics regardless whether the underlying conservative classical map is chaotic or not

(cf. Proposition 5.14). To recover useful information about the dynamics one needs to perform simultaneously the small noise and the semiclassical limit. It is thus clear that the notion of the quantum dissipation time is intrinsically of a semiclassical nature. It is, however, not enough to consider just 'a semiclassical limit', since the way the limits are taken here matters considerably.

The reason for this is the following: the classical dissipation time becomes larger and larger in the small-noise limit, but the correspondence between classical and quantum evolutions holds only up to a certain "breaking time", which diverges only in the semiclassical limit (see Chapter 4 for detailed discussion). Therefore, when seeking traces of chaoticity in quantum systems with classical chaotic counterparts one needs to consider sufficiently fast semiclassical limit.

On the other hand, one has to avoid falling into triviality waiting on another extreme of the problem. Namely, if the Planck constant is sent to zero too fast w.r.t. the size of the noise, the dissipation time can be shorter or even much shorter than the "breaking time" and the quantization effects disappear too quickly to be noticeable in the asymptotic behavior of the system.

The above situation is common to all approaches to quantum chaoticity in finite dimensional systems (see, e.g., results regarding the spectrum of noisy quantum propagators [28, 100], or the study of the decoherence rate and quantum dynamical entropy [6, 7, 20, 104, 57, 16]).

The problem one really needs to solve here is to determine the relation between, on the one hand, spatial scales represented by the the size of the noise and the magnitude of the Planck constant, and on the other hand, temporal scales, such as dissipation and "breaking" times, on which some traces of original classical chaoticity are still present in quantized systems.

The difficulty of the problem lies in the fact that in order to solve the problem one needs to control simultaneously the behavior of four different asymptotic

parameters (noise, the Planck constant, dissipation and breaking times).

It may be of some interest to remark that the problem resembles (to some extent) some problems occurring in numerical simulations of chaotic or turbulent dynamical systems. Any chaotic system develops, in a relatively short time, extremely complicated structures on smaller and smaller spatial scales. The quantization as well as numerical discretization inevitably imposes a finite resolution of the details of the phase space, the quantum 'mesh spacing' being constrained by the Heisenberg uncertainty principle $\Delta q \Delta p \gtrsim \hbar$. The introduction of a small amount of noise corresponds to numerical instabilities due to finite precision of any numerical computation (rounding errors).

Just as numerical approximations break down on sufficiently long time scales, one expects a similar breakdown of the quantum-classical correspondence. In the quantum framework, the corresponding time scale (the above "breaking time") is usually referred to as the *Ehrenfest* time τ_E (see Chapter 4). For chaotic systems, the first signs of discrepancy may appear around $\tau_E \approx \lambda^{-1} \ln(\hbar^{-1})$, where λ is the largest Lyapunov exponent (this is the earliest time scale on which the chaotic dynamics might develop structures beyond the reach of quantum phase-space resolution). Numerous works, both theoretical and numerical, have been devoted to study this phenomenon. Recently some rigorous results [22] have been obtained, describing the breakdown of the classical-quantum correspondence on such a time scale. However, this breakdown effectively occurs for a very particular class of maps and values of $\hbar$, and hence does not necessarily reflect the generic behavior (cf. Chapter 4).

For a given noise strength $\epsilon > 0$, a quantum dynamical system always resembles its classical counterpart if Planck's constant is small enough. This is reflected in Proposition 5.15, which states that for any canonical map and noise strength ϵ, the quantum dissipation time converges to its classical counterpart (see also Corollary 5.16). For a general map, it is more difficult to determine precisely a regime where

both $\epsilon,\ \hbar \to 0$, and such that classical and quantum dissipation times have the same asymptotics. We address this problem in the case of quantized toral automorphisms.

For symplectic toral automorphisms in arbitrary dimension, we prove (Theorem 5.17 and Corollary 5.18) this asymptotic correspondence between the classical and quantum dissipation times $\tau_c \approx \tau_q$ in the regime where $1 \gg \epsilon \geq C\hbar$. In this case, one indeed has $\tau_q \lesssim \tau_E$, which intuitively justifies the correspondence.

Around the "boundary" of this regime, that is for $\epsilon \sim C\hbar$, the noise strength is comparable with the "quantum mesh", and the dissipation and the Ehrenfest times are of the same order. It is important to stress that our methods allows us to prove that the asymptotics of the quantum and classical dissipation times coincide in an exact way, that is, including the prefactor in front of the logarithmic asymptotics (the "dissipation rate constant").

In case of quantum coarse graining (cf. def. (5.28)) we derive similar semiclassical result but under a little bit stronger assumption on the convergence rate for $\hbar$ - the existence of a positive β such that $\epsilon^{\beta}\hbar^{-1} > 1$.

We also investigate the opposite situation, when the classical dissipation time is longer than τ_E. In the situation where $\epsilon \ll \hbar^{1+\delta}$, we obtain (for ergodic toral automorphisms):

$$\tau_q \geq C\left(\frac{\hbar}{\epsilon}\right)^2 \gg \tau_c \sim \ln(\epsilon^{-1}) \gg |\ln \hbar| \sim \tau_E,$$

In this case, three different time ranges can be distinguished. Before τ_E, classical and quantum evolutions are identical (and do not dissipate). In the range $\tau_E \ll t \ll \tau_c$, the evolutions may differ, but neither dissipates yet. On the time scale $\tau_c \ll t \ll \tau_q$, the classical system dissipates, while the quantum one remains dissipationless until $t > \tau_q$.

This result shows in particular that after passing the Ehrenfest time the quantum and classical evolutions do not have to part from each other in an immediate and

complete fashion. For example, the difference between the systems is still not visible on that time scale if one restricts the observations only to dissipative properties of both systems.

To finish the description of our results we want to remark that a lot of important questions remain still open in this area and that the whole theory is rather in the initial stage of its development. The first problem is to generalize the above sharp estimates to nonlinear quantum maps (e.g. Anosov diffeomorphisms). This requires the application of different techniques (cf. [51]). In particular appropriate estimates are needed on the constant in Egorov theorem for such maps [25] (the problem is nonexistent in the linear case due to the fact that the semiclassical approximation is exact).

Also recent results on quantum dynamical entropy for finite dimensional systems [7, 16] seem to indicate that just like in the case of classical maps, it should be possible to establish a link between the dissipation rate constant and the quantum mechanical entropy of the system. It is still too early however to formulate any concrete conjectures (since the relation is not yet established for a sufficiently general class of classical systems). We discuss this problem briefly in Chapter 4.

We now conclude the Introduction with a few remarks regarding the organization of the material in this work. The dissertation is divided into five chapters and two appendixes (in the list below the descriptions do not necessary match the titles)

Chapter 1. Introduction.

Chapter 2. Definition and general properties of dissipation time.

Chapter 3. Dissipation time of classically chaotic systems on $\mathbb{T}^d$.

- Section 3.1 and 3.2 - Linear maps: toral automorphism and generalizations.

- Section 3.3 and 3.4 - Nonlinear maps: Anosov systems.

Chapter 4. Interludium.

Chapter 5. Dissipation time of quantum systems on $\mathbb{T}^d$.

- Section 5.1 - Weyl quantization on the Torus.

- Section 5.2 - Semiclassical analysis of dissipation time.

Appendix A. The dynamics of Cat maps.

Appendix B. Wigner transform.

Two chapters (1 and 4) are of expository character. The remaining chapters contain the main results of the work and constitute the original contribution to the field. Appendixes contain necessary technical background regarding two specific topics. Throughout the work attention was paid to ensure mathematical correctness and completeness. All results are presented with full proofs. For the convenience of the reader, technical proofs of secondary importance are collected at the end of each non-expository chapter in a separate section.

Part I

Classical Mechanics

Chapter 2

Dissipation time - definition, properties and general results

2.1 Setup and notation

2.1.1 Evolution operators

Let $(\mathbb{T}^d, \mathcal{B}(\mathbb{T}^d), m)$ denote the d-dimensional torus, equipped with its $\sigma-$field of Borel sets and the Lebesgue measure m. Let $F : \mathbb{T}^d \to \mathbb{T}^d$ be a map on the torus preserving the Lebesgue measure: for any set $B \in \mathcal{B}(\mathbb{T}^d)$ we have $m(F^{-1}(B)) = m(B)$. In general, F is not supposed to be invertible. In the following we call such a map 'volume preserving' with implicit reference to the Lebesgue measure.

The map F generates a discrete time dynamics on $\mathbb{T}^d$, which in terms of pathwise description can be represented by the forward trajectory $\{F^n(\boldsymbol{x}_0),\ n \in \mathbb{N}\}$ of any initial point (particle) $\boldsymbol{x}_0 \in \mathbb{T}^d$. However, instead of looking at the evolution of a single particle, one can consider the statistical description of the dynamics, that is the evolution of a density (more generally a measure) describing the initial statistical configuration of the system.

Let $\mathcal{M}(\mathbb{T}^d)$ denote the set of all Borel measures on $\mathbb{T}^d$. For any $\mu \in \mathcal{M}(\mathbb{T}^d)$ and

$f \in C^0(\mathbb{T}^d)$ we write

$$\mu(f) = \int_{\mathbb{T}^d} f(\boldsymbol{x}) d\mu(\boldsymbol{x}).$$

The map F induces a map F^* on $\mathcal{M}(\mathbb{T}^d)$ given by

$$(F^*\mu)(f) = \mu(f \circ F), \qquad \text{for all } f \in C^0(\mathbb{T}^d).$$

This map can also be defined as follows:

$$(F^*\mu)(B) = \mu(F^{-1}(B)), \qquad \text{for all } B \in \mathcal{B}(\mathbb{T}^d).$$

In particular if $\mu = \delta_{\boldsymbol{x}_0}$ then $F^*(\mu) = \delta_{F(\boldsymbol{x}_0)}$ and one recovers the pathwise description. If μ is absolutely continuous w.r.t. m, then $F^*(\mu)$ preserves this property (since the measure-preserving map F is nonsingular w.r.t. m, see [79, p.42]). The corresponding densities $g = \frac{d\mu}{dm} \in L^1(\mathbb{T}^d)$ are transformed by the Frobenius-Perron or transfer operator P_F [12]:

$$P_F\left(\frac{d\mu}{dm}\right) = \frac{d(F^*\mu)}{dm}.$$

If the map F is invertible, P_F is given explicitly by:

$$(P_F g)(\boldsymbol{x}) = (g \circ F^{-1})(\boldsymbol{x}) \frac{dF^*m}{dm}(\boldsymbol{x}) = g \circ F^{-1}(\boldsymbol{x}).$$

If the map F is differentiable, and the preimage set of $\boldsymbol{x}$ is finite for all $\boldsymbol{x}$, the Perron-Frobenius operator is given by

$$(P_F g)(\boldsymbol{x}) = \sum_{\boldsymbol{y} | F(\boldsymbol{y}) = \boldsymbol{x}} \frac{g(\boldsymbol{y})}{|J_F(\boldsymbol{y})|},$$

where $J_F(\boldsymbol{y})$ is the Jacobian of F at $\boldsymbol{y}$.

On the other hand one can consider the dual of the Frobenius-Perron operator, called the Koopman operator, which governs the evolution of observables $f \in L^\infty(\mathbb{T}^d)$ instead of that of densities $g \in L^1$. The Koopman operator U_F is defined as

$$U_F f = f \circ F. \tag{2.1}$$

Due to the nonseparability of the Banach space $L^\infty(\mathbb{T}^d)$, it is often more convenient to consider its closure in some weaker L^p norm, which yields larger (but separable) spaces of observables $L^p(\mathbb{T}^d)$. Here we will be mainly concerned with the space $L^2(\mathbb{T}^d)$ and its codimension-1 subspace of zero-mean functions

$$L_0^2(\mathbb{T}^d) = \{f \in L^2(\mathbb{T}^2) : m(f) = 0\}. \tag{2.2}$$

This subspace is obviously invariant under U_F and P_F, due to the assumption $F^*m = m$. Throughout the whole work, $\| \cdot \|$ will always refer to the L^2-norm (and corresponding operator norm) on $L_0^2(\mathbb{T}^d)$ (any other norm will carry an explicit subscript). For any measure-preserving map F, the operator U_F is an isometry on $L^2(\mathbb{T}^d)$ and $L_0^2(\mathbb{T}^d)$. When F is invertible, U_F is unitary on these spaces, and satisfies $U_F = P_F^{-1} = P_{F^{-1}}$.

Although just introduced operators will mostly be considered on $L_0^2(\mathbb{T}^d)$, we will need from time to time to act on some more specific spaces defined usually by certain regularity properties of functions belonging to them. Most typically these will be Hölder and Sobolev spaces, the definitions of which we now briefly recall and use this opportunity to fix the appropriate notation.

For any $m \in \mathbb{N}$, we denote by $C^m(\mathbb{T}^d)$ the space of m-times continuously differentiable functions, with the norm

$$\|f\|_{C^m} = \sum_{|\boldsymbol{\alpha}|_1 \le m} \|D^{\boldsymbol{\alpha}} f\|_\infty$$

(we use the norm $|\boldsymbol{\alpha}|_1 = \alpha_1 + \ldots + \alpha_d$ for the multiindex $\boldsymbol{\alpha} \in \mathbb{N}^d$). For any $s = m + \eta$ with $m = [s] \in \mathbb{N}$, $\eta \in (0, 1)$, let $C^s(\mathbb{T}^d)$ denote the space of C^m functions for which the m-derivatives are η-Hölder continuous; this space is equipped with the norm

$$\|f\|_{C^s} = \|f\|_{C^m} + \sum_{|\boldsymbol{\alpha}|_1 = m} \sup_{\boldsymbol{x} \ne \boldsymbol{y}} \frac{|D^{\boldsymbol{\alpha}} f(\boldsymbol{x}) - D^{\boldsymbol{\alpha}} f(\boldsymbol{y})|}{|\boldsymbol{x} - \boldsymbol{y}|^\eta}.$$

The Fourier transforms of functions $g \in L^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{T}^d)$ are defined as follows:

$$\forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad \hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} g(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{x}, \tag{2.3}$$

$$\forall \boldsymbol{k} \in \mathbb{Z}^d, \quad \hat{f}(\boldsymbol{k}) = \int_{\mathbb{T}^d} f(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{k}} d\boldsymbol{x} = \langle \boldsymbol{e_k}, f \rangle. \tag{2.4}$$

Above we used the Fourier modes on the torus $\boldsymbol{e_k}(\boldsymbol{x}) := e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{k}}$. For any $s \geq 0$, we denote by $H^s(\mathbb{T}^d)$ and $H^s(\mathbb{R}^d)$ the Sobolev spaces of s-times weakly differentiable L^2-functions equipped with the norms $\|\cdot\|_{H^s}$ defined respectively by

$$\|g\|^2_{H^s(\mathbb{R}^d)} = \int_{\boldsymbol{\xi}\in\mathbb{R}^d} (1+|\boldsymbol{\xi}|^2)^s |\hat{g}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi},$$

$$\|f\|^2_{H^s(\mathbb{T}^d)} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (1+|\boldsymbol{k}|^2)^s |\hat{f}(\boldsymbol{k})|^2.$$

Finally, for any of these spaces, adding the subscript 0 will mean that we consider the (U_F-invariant) subspace of functions with zero average, e.g. $C_0^j(\mathbb{T}^d) = \{f \in C^j(\mathbb{T}^d),\ m(f) = 0\}$.

2.1.2 Noise operator

To construct the noise operator we first define the *noise generating density* i.e. an arbitrary probability density function $g \in L^1(\mathbb{R}^d)$ symmetric w.r.t. the origin $g(\boldsymbol{x}) = g(-\boldsymbol{x})$. The *noise width* (or *noise level*) will be given by a single nonnegative parameter, which we call ϵ. To each $\epsilon > 0$ there corresponds the noise kernel on $\mathbb{R}^d$:

$$g_\epsilon(\boldsymbol{x}) = \frac{1}{\epsilon^d} g\left(\frac{\boldsymbol{x}}{\epsilon}\right),$$

with the convention that $g_0 = \delta_0$. The noise kernel on the torus is obtained by periodizing g_ϵ, yielding the periodic kernel

$$\tilde{g}_\epsilon(\boldsymbol{x}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} g_\epsilon(\boldsymbol{x}+\boldsymbol{n}). \tag{2.5}$$

We remark that the Fourier transform of $\tilde{g}_\epsilon$ is related to that of g by the identities $\hat{\tilde{g}}_\epsilon(\boldsymbol{k}) = \hat{g}_\epsilon(\boldsymbol{k}) = \hat{g}(\epsilon\boldsymbol{k})$.

The action of the noise operator G_ϵ on any function $f \in L^2_0(\mathbb{T}^d)$ is defined by the convolution:

$$G_\epsilon f \quad = \quad \tilde{g}_\epsilon * f.$$

As a convolution operator with kernel from L^1, G_ϵ is compact on $L^2_0(\mathbb{T}^d)$ (if g is square-integrable, G_ϵ is Hilbert-Schmidt). The Fourier modes $\{\boldsymbol{e_k}\}_{0\neq \boldsymbol{k}\in\mathbb{Z}^d}$ form an orthonormal basis of eigenvectors of G_ϵ, yielding the following spectral decomposition:

$$\forall f \in L^2_0(\mathbb{T}^d), \quad G_\epsilon f = \sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \hat{g}(\epsilon \boldsymbol{k})\langle \boldsymbol{e_k}, f\rangle\, \boldsymbol{e_k}. \tag{2.6}$$

This formula shows that the eigenvalue associated with $\boldsymbol{e_k}$ is $\hat{g}(\epsilon\boldsymbol{k})$. Since g is a symmetric function, this eigenvalue is real, so that G_ϵ is a self-adjoint operator. Its spectral radius $r_{sp}(G_\epsilon)$ is therefore given by

$$r_{sp}(G_\epsilon) = \|G_\epsilon\| = \sup_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} |\hat{g}(\epsilon\boldsymbol{k})|. \tag{2.7}$$

Since the density g is positive, $\hat{g}$ attains its maximum $\hat{g}(0) = 1$ at the origin and nowhere else. Besides, because $g \in L^1(\mathbb{R}^d)$, $\hat{g}$ is a continuous function vanishing at infinity. As a result, for small enough $\epsilon > 0$, the supremum on the RHS of (2.7) is reached at some point $\epsilon\boldsymbol{k}$ close to the origin, and this maximum is strictly smaller than 1. This shows that the operator G_ϵ is strictly contracting on $L^2_0(\mathbb{T}^d)$:

$$\forall \epsilon > 0, \quad \|G_\epsilon\| = r_{sp}(G_\epsilon) < 1. \tag{2.8}$$

In the next section we study this noise operator more precisely, starting from appropriate assumptions on the noise generating density.

2.1.3 Noise kernel estimates

In this subsection we present some estimates regarding the noise operator. These estimates will later play a crucial role in the derivation of the asymptotics of the dissipation time.

We will be interested in the behavior of the system in the limit of small noise, that is the limit $\epsilon \to 0$. It will hence be useful to introduce the following asymptotic notation. Given two variables a_ϵ, b_ϵ depending on $\epsilon > 0$, we write

$$a_\epsilon \lesssim b_\epsilon \text{ if } \limsup_{\epsilon \to 0} \frac{a_\epsilon}{b_\epsilon} < \infty, \tag{2.9}$$

$$a_\epsilon \approx b_\epsilon \text{ if } \lim_{\epsilon \to 0} \frac{a_\epsilon}{b_\epsilon} = 1, \tag{2.10}$$

$$a_\epsilon \sim b_\epsilon \text{ if } a_\epsilon \lesssim b_\epsilon \text{ and } b_\epsilon \lesssim a_\epsilon. \tag{2.11}$$

In order to obtain interesting estimates on the noise operator G_ϵ, it will be necessary to impose some additional conditions on its generating density g, regarding e.g. its rate of decay at infinity, or the behavior of its Fourier transform near the origin.

The weakest condition we are going to impose is the existence of some positive moment of g, by which we mean that for some $\alpha \in (0, 2]$,

$$M_\alpha = \int_{\mathbb{R}^d} |\boldsymbol{x}|^\alpha g(\boldsymbol{x}) d\boldsymbol{x} < \infty \tag{2.12}$$

(we take the length $|\boldsymbol{x}| = (x_1^2 + \ldots + x_d^2)^{1/2}$ on $\mathbb{R}^d$). This condition implies the following properties of the Fourier transform $\hat{g}$ (proved in Section 2.8)

Lemma 2.1 *For any $\alpha \in (0, 2]$ there exists a universal constant C_α such that, if a normalized density g satisfies (2.12), then the following inequalities hold:*

$$\forall \boldsymbol{\xi} \in \mathbb{R}^d, \quad 0 \leq 1 - \hat{g}(\boldsymbol{\xi}) \leq C_\alpha M_\alpha |\boldsymbol{\xi}|^\alpha. \tag{2.13}$$

If (2.12) holds with $\alpha = 2$, we have the more precise information:

$$1 - \hat{g}(\boldsymbol{\xi}) \sim |\boldsymbol{\xi}|^2 \quad \textit{in the limit } \boldsymbol{\xi} \to 0.$$

In the case $\alpha < 2$, we will sometimes assume a stronger property than (2.13), namely that

$$1 - \hat{g}(\boldsymbol{\xi}) \sim |\boldsymbol{\xi}|^\alpha \quad \text{in the limit } \boldsymbol{\xi} \to 0. \tag{2.14}$$

Note that this behavior implies a uniform bound $1 - \hat{g}(\boldsymbol{\xi}) \leq C|\boldsymbol{\xi}|^{\gamma}$ for any $\gamma \leq \alpha$ and C independent of γ.

Typical examples of noise kernels satisfying (2.14) include the Gaussian kernel and more general symmetric $\alpha-$stable kernels [119, p.152] defined for $\alpha \in (0, 2]$:

$$g_{\epsilon,\alpha}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} e^{-(\boldsymbol{Q}(\epsilon\boldsymbol{k}))^{\alpha/2}} \boldsymbol{e}_{\boldsymbol{k}}(\boldsymbol{x}), \tag{2.15}$$

where $\boldsymbol{Q}$ denotes an arbitrary positive definite quadratic form. For the values of α indicated, the function $g_{\epsilon,\alpha}(x)$ is positive on $\mathbb{R}^d$.

In view of Eq.(2.7), the properties (2.12) or (2.14) determine the rate at which G_ϵ contracts on $L^2_0(\mathbb{T}^d)$. For instance, (2.14) implies that in the limit $\epsilon \to 0$,

$$1 - \|G_\epsilon\| \sim \epsilon^\alpha. \tag{2.16}$$

The following proposition describes the effect of the noise on various types of observables, in the limit of small noise level. The proof is given in Section 2.8.

Proposition 2.2 *i) For any noise generating density $g \in L^1(\mathbb{R}^d)$ and any observable $f \in L^2_0(\mathbb{T}^d)$, one has*

$$\|G_\epsilon f - f\| \xrightarrow{\epsilon\to 0} 0. \tag{2.17}$$

To obtain information on the speed of convergence, we need to impose constraints on both the noise kernel and the observable.

ii) If for some $\alpha \in (0, 2]$ the kernel g satisfies (2.12) or (2.14), then for any $\gamma > 0$ there exists a constant $C > 0$ such that for any observable $f \in H^\gamma(\mathbb{T}^d)$,

$$\|G_\epsilon f - f\| \leq C\epsilon^{\gamma\wedge\alpha}\|f\|_{H^{\gamma\wedge\alpha}}, \tag{2.18}$$

where $\gamma \wedge \alpha := \min\{\gamma, \alpha\}$. If $f \in C^1(\mathbb{T}^d)$, the above upper bound can be replaced by

$$\|G_\epsilon f - f\| \leq C\epsilon^{1\wedge\alpha}\|\nabla f\| \leq C\epsilon^{1\wedge\alpha}\|\nabla f\|_\infty. \tag{2.19}$$

Using the noise operator, we are now in position to define the noisy (resp. the coarse-grained) dynamics generated by a measure-preserving map F.

2.1.4 Noisy evolution operators

The noisy evolution through the map F is constructed by successive application of the Koopman operator U_F and the noise operator G_ϵ. The noisy dynamics is then generated by taking powers of the *noisy propagator*

$$T_\epsilon = G_\epsilon U_F.$$

In general, the operator T_ϵ is not normal, but satisfies $r_{sp}(T_\epsilon) \leq \|T_\epsilon\| = \|G_\epsilon\|$. We will also consider a *coarse-grained dynamics* defined by the application of the noise kernel only at the beginning and at the end of the evolution. Hence we define the following family of operators:

$$\tilde{T}_\epsilon^{(n)} = G_\epsilon U_F^n G_\epsilon, \quad n \in \mathbb{N}.$$

In view of the contracting properties of G_ϵ, the inequalities $\|T_\epsilon^n\| \leq \|G_\epsilon\|^n$, $\|\tilde{T}_\epsilon^{(n)}\| \leq \|G_\epsilon\|^2$ imply that both noisy and coarse-grained operators are strictly contracting on $L_0^2(\mathbb{T}^d)$.

2.2 General definition of dissipation time

Once the notation has been set up, we can pass to the precise definition of the dissipation time. We prefer to start, however, with some remarks regarding the motivation. In particular we briefly recall the original, continuous-time physical setting considered in [48], where the notion had been introduced for the first time. We then generalize the original definition to an abstract, discrete-time setting and after discussing some of the most basic properties of this notion we conclude the section with another physical interpretation, this time coming from statistical physics and expressed in terms of Boltzmann-Gibbs Entropy.

2.2.1 Motivation

In order to gain a bit of physical intuition it is useful to consider the original problem described in [48], where the notion of dissipation time was introduced in the context of continuous-time dynamical system of a passive tracer in a fluid flow. The flow is prescribed in terms of given *periodic*, *incompressible* velocity filed and the path of the tracer is randomly perturbed by collisions with fluid particles. The latter phenomena being modeled by standard Brownian motion. In pathwise description the evolution of the tracer is given by the following Langevin-type equation

$$d\mathbf{x}^{\varepsilon}(t) = \mathbf{u}(\mathbf{x}^{\varepsilon}(t))dt + \sqrt{\varepsilon}d\mathbf{w}(t), \quad \nabla \cdot \mathbf{u}(\mathbf{x}) = 0,$$

where $\mathbf{w}$ stands for the standard Brownian motion. In Statistical description, the dynamics on the densities is given by the corresponding Fokker-Planck (advection-diffusion) equation

$$\frac{\partial \rho}{\partial t} = \mathbf{u} \cdot \nabla \rho + \frac{\varepsilon}{2}\Delta \rho, \quad \nabla \cdot \mathbf{u} = 0, \tag{2.20}$$

Let us note that unperturbed skew-symmetric operator $\mathbf{u} \cdot \nabla$ generates an unitary (conservative) group $U^t = e^{t\mathbf{u}\cdot\nabla}$, which corresponds (for $t = 1$) to our conservative Koopman operator U_F for some F.

Now, the perturbed generator $\mathcal{L}_\epsilon = \mathbf{u}\cdot\nabla + \frac{\varepsilon}{2}\Delta$ gives rise to a semi-group of contractions $P_\epsilon^t = e^{t\mathcal{L}_\epsilon}$, which heuristically can be thought of as continuous time-1 counterpart of the generator $T_\epsilon = G_\epsilon U_F$ of our discrete-time noisy dynamics in case of Gaussian noise (we say 'heuristically', because due to noncommutativity of both terms in $\mathcal{L}_\epsilon$ there is no obvious way of associating T_ϵ with P_ϵ for general velocity fields $\mathbf{u}$).

In order to predict long time behavior of the tracer and in particular the influence of the noise on its trajectory, one is interested in determining the speed of contraction of the semigroup P_ϵ. In [48] (see also more recent version in [49]) the following time scale t_{diss} defined by equation $\|P_\epsilon^{t_{diss}}\| = 1/2$ was suggested for consideration regarding

this problem and termed as *dissipation time*. Determining the asymptotics of t_{diss} (in continuous-time setting) proved however to be exceedingly difficult and except for some very special and simple cases (e.g. cellular flow considered in [49]) the problem has not been solved up to date. Also there is no known example of non-trivially short dissipation time (i.e. logarithmic in ϵ^{-1}) in continuous setting. The main difficulty, as was already observed in [48], lies in the fact that to get this result one would need to consider fully chaotic system, while it is very difficult to construct one i.e. to write down a simple differential equation which would exhibit fully chaotic behavior. On the other hand there is no problem in constructing fully chaotic discrete-time systems and in fact the ergodic theory literature abounds in such examples. This situation provided natural motivation for the generalization of the notion of the dissipation time to discrete-time systems.

2.2.2 The definition

In its general form the classical dissipation time $\tau_c(p)$ for discrete-time noisy dynamics T_ϵ is defined in terms of the norm $\|\cdot\|_{p,0}$ on the space $L^p_0(\mathbb{T}^d)$ and w.r.t. a threshold $\eta \in (0,1)$.

Definition 2.3 *Let T_ϵ denote discrete-time noisy dynamics. We define*

$$\tau_c(p,\eta) := \min\{n \in \mathbb{Z}_+ : \|T^n_\epsilon\|_{p,0} < \eta\}, \quad 1 \le p \le \infty. \tag{2.21}$$

The fact that T_ϵ acts on every $L^p_0(\mathbb{T}^d)$ as a strict contraction ensures existence and uniqueness of $\tau_c(p,\eta)$, for each $\eta \in (0,1)$.

Figure 2.1 illustrates the definition.

We need to show that the value of the threshold η in (2.21) does not affect the order of divergence of $\tau_c(p,\eta)$, as ϵ tends to zero.

Proposition 2.4 *For any $0 < \tilde{\eta}, \eta < 1$, $\tau_c(p,\tilde{\eta}) \sim \tau_c(p,\eta)$.*

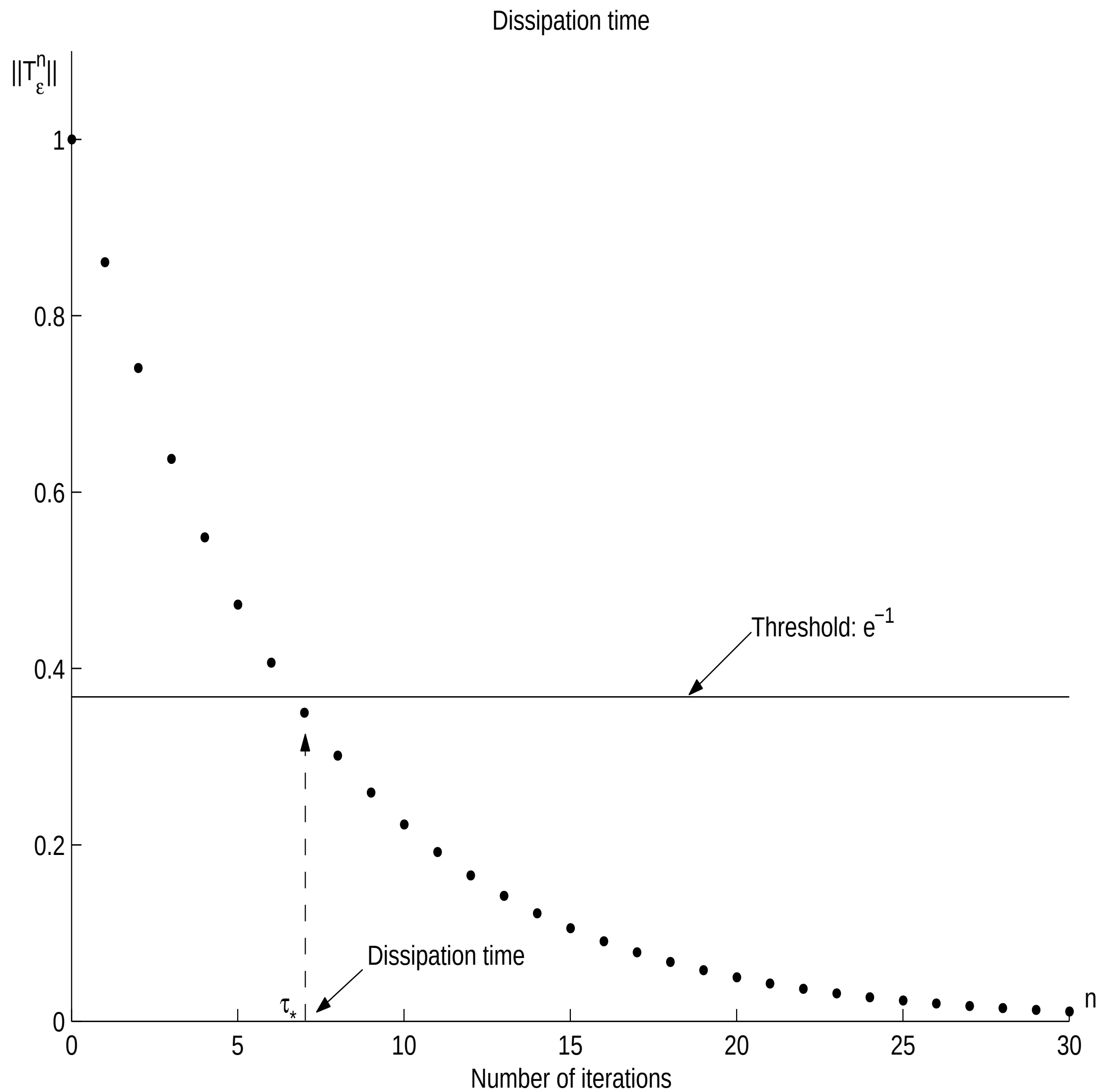

Figure 2.1: Definition of the dissipation time

Proof. Assume $0 < \tilde{\eta} < \eta < 1$. Obviously $\tau_c(p, \tilde{\eta}) \geq \tau_c(p, \eta)$. On the other hand let k be a positive integer such that $\eta^k < \tilde{\eta}$. Then

$$\|T_\epsilon^{\tau_c(p,\eta)}\|_{p,0} < \eta \Rightarrow \|T_\epsilon^{k\tau_c(p)}\|_{p,0} < \eta^k < \tilde{\eta}.$$

Hence $k\tau_c(p, \eta) \geq \tau_c(p, \tilde{\eta})$, which implies $\tau_c(p, \eta) \sim \tau_c(p, \tilde{\eta})$. ■

Following the argument of [111] one can use the Riesz convexity theorem to establish also the asymptotic equivalence of the $\tau_c(p)$, for all $1 < p < \infty$ (to alleviate the notation we drop η).

Proposition 2.5 *i) For any* $1 < q, p < \infty$, $\tau_c(q) \sim \tau_c(p)$.
ii) For any $1 < p < \infty$, $\tau_c(p) \lesssim \tau_c(1)$ *and* $\tau_c(p) \lesssim \tau_c(\infty)$.

We postpone a technical proof of this proposition to Section 2.8.

In view of the above results, instead of working in general setting, one can choose some convenient values of p and η and perform, without any loss of generality, all necessary asymptotic calculations in one notationally simplified setting. We will usually choose $p = 2$ and $\eta = e^{-1}$ for computational convenience. Following this choice we introduce the convention that $\tau_c(p) := \tau_c(p, e^{-1})$ and $\tau_c := \tau_c(2, e^{-1})$. The obvious dependence of the dissipation time on ϵ will always be implicitly assumed but rarely explicitly denoted.

The corresponding dissipation time for a coarse-grained dynamics is defined in fully analogous way and denoted respectively by $\tilde{n}_c(p, \eta)$, $\tilde{n}_c(p)$ and $\tilde{n}_c$. In particular

$$\tilde{\tau}_c := \min\{n \in \mathbb{N} : \|\tilde{T}_\epsilon^{(n)}\| < e^{-1}\}. \tag{2.22}$$

We note that the dissipation time does not depend on whether the dynamics is applied to densities (i.e. by the Frobenius-Perron operator) or to observables (by the Koopman operator). Indeed, the norm of an operator equals the norm of its adjoint [125, p.195], so that

$$\|\tilde{T}_\epsilon^{(n)}\| = \|G_\epsilon U_F^n G_\epsilon\| = \|(G_\epsilon U_F^n G_\epsilon)^*\| = \|G_\epsilon P_F^n G_\epsilon\|,$$

and similarly for the noisy operator T_ϵ. In particular, for invertible maps the dissipation time does not depend on the direction of time.

As mentioned in the Introduction, we will distinguish two qualitatively different asymptotic behaviors of dissipation time in the limit $\epsilon \to 0$. We say that the operator T_ϵ (or the map F associated with it) respectively has

I) *simple* or *power-law* dissipation time if there exists $\beta > 0$ such that

$$\tau_c \sim 1/\epsilon^\beta,$$

II) *fast* or *logarithmic* dissipation time if

$$\tau_c \sim \ln(1/\epsilon).$$

We will also speak about *slow* dissipation time whenever there exists some $\beta > 0$ s.t.

$$\tau_c \gtrsim 1/\epsilon^\beta.$$

In case of logarithmic dissipation time, the dissipation rate constant R_c, when it exists, is defined as

$$R_c = \lim_{\epsilon \to 0} \frac{\tau_c}{\ln(1/\epsilon)}. \tag{2.23}$$

A similar terminology will be applied to the coarse-grained dissipation time $\tilde{\tau}_c$.

2.2.3 Physical interpretation via Boltzmann-Gibbs entropy

In this section we briefly discuss the connection between dissipation time and Boltzmann-Gibbs entropy. The results formalize an intuitive physical interpretation of the dissipation time outlined in the Introduction.

First we note that on the scales exceeding τ_c, the Boltzmann-Gibbs entropy approaches its maximal equilibrium value (i.e. 0) as can be seen from the following

simple estimate (cf. [79]). Let us first restrict considerations to bounded initial states, i.e., $f \geq 0, f \in L^\infty$ and $\|f\|_1 = 1$. Let

$$\eta(u) = \begin{cases} -u \ln u, & u > 0 \\ 0, & u = 0 \end{cases}$$

and let $D_n = \{\boldsymbol{x} \in \mathbb{T}^d : 1 \leq T_\epsilon^n f\}$. On one hand, we have

$$\begin{aligned} & \left| \int_{D_n} \eta(T_\epsilon^n f(\boldsymbol{x})) d\boldsymbol{x} \right| \\ \leq & \int_{D_n} \left| \int_1^{T_\epsilon^n f(\boldsymbol{x})} \frac{d\eta(u)}{du} du \right| d\boldsymbol{x} \\ \leq & \sup_{1 \leq u \leq \|T_\epsilon^n f\|_\infty} (1 + \ln u) \int_{D_n} |T_\epsilon^n f(\boldsymbol{x}) - 1| d\boldsymbol{x} \\ \leq & (1 + \ln \|T_\epsilon^n f\|_\infty) \|T_\epsilon^n f - 1\|_1 \\ \leq & (1 + \ln \|f\|_\infty) \|T_\epsilon^n f - 1\|_1. \end{aligned} \tag{2.24}$$

On the other hand, we have

$$0 \geq \int_{\mathbb{T}^d} \eta(T_\epsilon^n f(\boldsymbol{x})) d\boldsymbol{x} \geq \int_{D_n} \eta(T_\epsilon^n f(\boldsymbol{x})) d\boldsymbol{x}.$$

In view of the inclusion relation: $L^\infty(\mathbb{T}^d) \subset L^2(\mathbb{T}^d) \subset L^1(\mathbb{T}^d)$, we then obtain that for $n \gg \tau_c$

$$\sup_{f \geq 0, \|f\|_\infty \leq c} \left| \int_{\mathbb{T}^d} \eta(T_\epsilon^n f(\boldsymbol{x})) d\boldsymbol{x} \right| \xrightarrow{\epsilon \downarrow 0} 0, \quad \forall c > 0.$$

For unbounded initial states, we note that, by Young's inequality,

$$\|T_\epsilon^n f\|_\infty \leq \|T_\epsilon f\|_\infty \leq \|g_\epsilon\|_\infty \|f\|_1 = \|g_\epsilon\|_\infty$$

from which we have, instead of (2.24), the following estimate

$$\left| \int_{D_n} \eta(T_\epsilon^n f(\boldsymbol{x})) d\boldsymbol{x} \right| \leq (1 + \ln \|g_\epsilon\|_\infty) \|T_\epsilon^n f - 1\|_1.$$

where in view of (2.16)

$$\ln \|g_\epsilon\|_\infty \sim \ln(1/\epsilon).$$

Therefore for sufficiently fast diverging $n \gg \tau_c(1)$ such that

$$\ln(1/\epsilon)\|T_\epsilon^n(f-1)\|_{1,0} \xrightarrow{\epsilon\downarrow 0} 0 \tag{2.25}$$

one obtains

$$\sup_{f\geq 0, \|f\|_1=1} \left| \int_{\mathbb{T}^d} \eta(T_\epsilon^n f(\boldsymbol{x}))d\boldsymbol{x} \right| \xrightarrow{\epsilon\downarrow 0} 0.$$

The condition (2.25) typically results in a slightly longer time scale than $\tau_c(1)$.

On the other hand, we can bound the L_1 distance between the probability density function f and the Lebesgue measure by their relative entropy via Csiszár's inequality [38]

$$\int_{\mathbb{T}^d} |f(\boldsymbol{x}) - g(\boldsymbol{x})| d\boldsymbol{x} \leq \sqrt{2\int_{\mathbb{T}^d} f(\boldsymbol{x}) \ln\left(f(\boldsymbol{x})/g(\boldsymbol{x})\right) d\boldsymbol{x}}$$

with $g(\boldsymbol{x}) = 1$. We see immediately that the decay rate of

$$\sup_{f\geq 0, \|f\|_1=1} \left| \int_{\mathbb{T}^d} \eta(T_\epsilon^n f(\boldsymbol{x}))d\boldsymbol{x} \right|$$

provides an estimate for $\tau_c(1)$ and, consequently, for $\tau_c(p), p \in (1,\infty)$.

2.3 Dissipation time and spectral analysis

In this section we investigate the connection between the dissipation time of the noisy propagator T_ϵ and its pseudospectrum together with some spectral properties of U_F and G_ϵ. All the operators considered in this section are defined on $L_0^2(\mathbb{T}^d)$. In the framework of continuous-time dynamics, some connections have recently been obtained between, on one side, the pseudospectrum of the (non-selfadjoint) generator A, and on the other side, the norm of the evolution operator e^{tA} [39]. We consider complementary, discrete-time setting, which allows for generalizations and more transparent proofs. We start with the definition of the pseudospectrum, and then derive general abstract lower and upper bounds for the dissipation time. In the following sections we will apply these results to determine the asymptotics of the dissipation

time under some dynamical assumptions regarding the underlying conservative maps (e.g. lack of weak-mixing).

2.3.1 Pseudospectrum

In this short subsection we define the pseudospectrum of a bounded operator [123] and state some of its properties.

Definition 2.6 *Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$ (we note $T \in \mathcal{L}(\mathcal{H})$). For any $\delta > 0$, the δ-pseudospectrum of T (denoted by $\sigma_\delta(T)$) can be defined in the following three equivalent ways:*

$$\begin{aligned}
(I) &\quad \sigma_\delta(T) = \{\lambda \in \mathbb{C} : \|(\lambda - T)^{-1}\| \geq \delta^{-1}\}, \\
(II) &\quad \sigma_\delta(T) = \{\lambda \in \mathbb{C} : \exists v \in \mathcal{H},\ \|v\| = 1, \quad \|(T - \lambda)v\| \leq \delta\}, \\
(III) &\quad \sigma_\delta(T) = \{\lambda \in \mathbb{C} : \exists B \in \mathcal{L}(\mathcal{H}),\ \|B\| \leq \delta, \quad \lambda \in \sigma(T + B)\}.
\end{aligned}$$

We will apply these definitions to the operator T_ϵ. For brevity, the resolvent of this operator will be denoted by $R_\epsilon(\lambda) = (\lambda - T_\epsilon)^{-1}$. We call S^r the circle $\{\lambda \in \mathbb{C} : |\lambda| = r\}$ in the complex plane, and define the following *pseudospectrum distance function*:

$$d_\epsilon(r) := \inf\{\delta > 0 : \sigma_\delta(T_\epsilon) \cap S^r \neq \emptyset\}.$$

From the definition (I) of the pseudospectrum, one easily shows that this distance is also given by

$$d_\epsilon^{-1}(r) = \sup_{|\lambda| = r} \|R_\epsilon(\lambda)\|. \tag{2.26}$$

We have the following property (proved in Section 2.8):

Proposition 2.7 *For any isometry U and noise generating function g, one has*

$$d_\epsilon(1) \xrightarrow{\epsilon \to 0} 0. \tag{2.27}$$

This means that for any fixed $\delta > 0$, the pseudospectrum $\sigma_\delta(T_\epsilon)$ will intersect the unit circle for small enough ϵ.

2.3.2 General bounds for the dissipation time

In this section we consider both fully noisy and coarse grained dynamics. We start with 'non-finiteness' results.

Proposition 2.8 *For any measure-preserving map F and any noise generating function g, both fully noisy and coarse-grained dissipation times diverge in the small-noise limit $\epsilon \to 0$.*

Proof. We skip the subscript F to alleviate the notation. We only use the fact that $U = U_F$ is an isometry. We start with the full noisy case and prove by induction the following strong convergence of operators

$$\forall f \in L_0^2(\mathbb{T}^d), \quad \forall n \in \mathbb{N}, \qquad \|T_\epsilon^n f - U^n f\| \xrightarrow{\epsilon \to 0} 0.$$

From Proposition 2.2*i)*, this limit holds in the case $n = 1$. Let us assume it holds at the rank $n-1$. Then we write

$$T_\epsilon^n f = U T_\epsilon^{n-1} f + (G_\epsilon - I) U T_\epsilon^{n-1} f.$$

From the inductive hypothesis, $T_\epsilon^{n-1} f \xrightarrow{\epsilon \to 0} U^{n-1} f$, so that the first term on the RHS converges to $U^n f$. Applying Proposition 2.2*i)* to the function $U^n f$, we see that the second term vanishes in the limit $\epsilon \to 0$. From the isometry of U, we obtain that for any $n > 0$, $\|T_\epsilon^n\| \xrightarrow{\epsilon \to 0} 1$, so that $\tau_c \xrightarrow{\epsilon \to 0} \infty$. In coarse-grained version, similarly as above, we have

$$\begin{aligned} \|\tilde{T}_\epsilon^n f - U^n f\| &= \|G_\epsilon U_F^n (G_\epsilon - I) f + (G_\epsilon - I) U^n f\| \\ &\leq \|(G_\epsilon - I) f\| + \|(G_\epsilon - I) U^n f\| \to 0. \qquad \blacksquare \end{aligned}$$

Now we pass to abstract spectral bounds.

Theorem 2.9 *For any isometric operator U on $L_0^2(\mathbb{T}^d)$ and noise operator G_ϵ, the dissipation time of the noisy evolution operator $T_\epsilon = G_\epsilon U$ satisfies the following esti-*

mates:

$$\frac{1-e^{-1}}{d_\epsilon(1)} \le \tau_c \le \frac{1}{|\ln(\|G_\epsilon\|)|} + 1, \tag{2.28}$$

$$\tau_c \le \inf_{r_{sp}(T_\epsilon)<r<1} \frac{1}{|\ln(r)|} \ln\left(\frac{e}{d_\epsilon(r)}\right). \tag{2.29}$$

We notice that the first upper bound does not depend on U at all, but only on the noise. Using the estimate (2.16), we obtain the following obvious corollary:

Corollary 2.10 *If the noise generating density satisfies the estimate* (2.14) *for some* $\alpha \in (0,2]$*, then for any measure-preserving map* F *the noisy dissipation time is bounded from above as follows* $\tau_c \lesssim \epsilon^{-\alpha}$.

Proof of Theorem 2.9.

1. Lower bound

We use the following series expansion of the resolvent [125, p.211] valid for any $|\lambda| > r_{sp}(T_\epsilon)$:

$$R_\epsilon(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n-1} T_\epsilon^n. \tag{2.30}$$

Considering that $r_{sp}(T_\epsilon) \le \|G_\epsilon\| < 1$, we may take $|\lambda| = 1$, and cut this sum into two parts:

$$R_\epsilon(\lambda) = \sum_{n=0}^{\tau_c-1} \lambda^{-n-1} T_\epsilon^n + \lambda^{-\tau_c} T_\epsilon^{\tau_c} R_\epsilon(\lambda).$$

Taking norms and applying the triangle inequality, we get

$$\begin{aligned} \|R_\epsilon(\lambda)\| &\le \|\sum_{n=0}^{\tau_c-1} \lambda^{-n-1} T_\epsilon^n\| + |\lambda|^{-\tau_c} \|T_\epsilon^{\tau_c}\| \|R_\epsilon(\lambda)\| \\ &\le \tau_c + e^{-1}\|R_\epsilon(\lambda)\| \\ \Longrightarrow \|R_\epsilon(\lambda)\|(1-e^{-1}) &\le \tau_c. \end{aligned}$$

Taking the supremum over $\lambda \in S_1$ yields the lower bound.

2. Upper bounds

To get both upper bounds, we use the following trivial lemma.

Lemma 2.11 *Assume that (for some value of ϵ) the powers of T_ϵ satisfy*

$$\forall n \in \mathbb{N}, \quad \|T_\epsilon^n\| \leq \Gamma(n),$$

where the function $\Gamma(n)$ is strictly decreasing, and $\Gamma(n) \xrightarrow{n\to\infty} 0$. Then the dissipation time is bounded from above by

$$\tau_c \leq \Gamma^{(-1)}(e^{-1}) + 1,$$

where $\Gamma^{(-1)}$ is the inverse function of Γ. In particular, for the geometric decay $\Gamma(n) = Cr^n$ with $r \in (0,1)$, $C \geq 1$, one obtains $\tau_c \leq \frac{\ln(eC)}{|\ln r|} + 1$.

The upper bound in Eq. (2.28) comes from the obvious estimate

$$\|T_\epsilon^n\| \leq \|G_\epsilon\|^n,$$

on which we apply the lemma with $C = 1$, $r = \|G_\epsilon\|$.

To prove the second upper bound, we use the representation of T_ϵ^n in terms of the resolvent:

$$T_\epsilon^n = \frac{1}{2\pi i} \int_{S^r} \lambda^n R_\epsilon(\lambda) d\lambda$$

valid for any $r > r_{sp}(T_\epsilon)$. Thus for all $r \in (r_{sp}(T_\epsilon), 1)$, one has

$$\|T_\epsilon^n\| \leq \frac{1}{2\pi} \int_{S^r} |\lambda|^n \|R_\epsilon(\lambda)\| |d\lambda| \leq \sup_{|\lambda|=r} \|R_\epsilon(\lambda)\| r^{n+1} = \frac{1}{d_\epsilon(r)} r^{n+1}.$$

We then apply Lemma 2.11 on the geometric decay for any radius $r_{sp}(T_\epsilon) < r < 1$, with $C = \frac{r}{d_\epsilon(r)} \geq 1$.

$\blacksquare$

2.4 Dissipation time of not weakly-mixing maps

In order to better control the growth of τ_c, we need more precise information on the noise and the dynamics. In the present section, we restrict ourselves to the dynamical property of weak-mixing. We recall [36] that the map F is ergodic (resp. weakly-mixing) iff 1 is not an eigenvalue of U_F (resp. iff U_F has no eigenvalue) on $L^2_0(\mathbb{T}^d)$. We now use Theorem 2.9 in the case where $U = U_F$ is the Koopman operator for some measure-preserving map F on $\mathbb{T}^d$ to establish the following important result.

Theorem 2.12 *Assume that the noise generating density g satisfies the estimates* (2.12) *or* (2.14) *with exponent $\alpha \in (0,2]$. If F is not weakly-mixing and at least one eigenfunction of U_F belongs to $H^\gamma(\mathbb{T}^d)$ for some $\gamma > 0$, then T_ϵ has slow dissipation time:*

$$\epsilon^{-(\alpha\wedge\gamma)} \lesssim \tau_c.$$

Proof. Let $h \in H^\gamma(\mathbb{T}^d)$ be a normalized eigenfunction of U_F with eigenvalue λ. Applying Proposition 2.2 *ii)*, we get

$$\|(\lambda - T_\epsilon)h\| = \|(I - G_\epsilon)h\| \leq K\epsilon^{\gamma\wedge\alpha}$$

for some constant $K > 0$ depending on g and h. This implies that $\|R_\epsilon(\lambda)\| \geq \frac{1}{K\epsilon^{\gamma\wedge\alpha}}$, therefore taking the supremum over $|\lambda| = 1$ yields $d_\epsilon(1)^{-1} \geq \frac{1}{K\epsilon^{\gamma\wedge\alpha}}$. The lower bound in Theorem 2.9 then implies

$$\frac{1-e^{-1}}{K\epsilon^{\gamma\wedge\alpha}} \leq \tau_c. \qquad \blacksquare \tag{2.31}$$

Remark 2.13 *Recall that if g satisfies* (2.14) *with exponent α, then the dissipation time is also bounded from above, as shown in Corollary 2.10. If one eigenfunction h has regularity H^γ with $\gamma \geq \alpha$, then both results imply that the dissipation is* simple, *with exponent α.*

Remark 2.14 *The above results can be stated in more general form: U_F does not need to be a Koopman operator associated with a map F. The result holds true for any isometric operator U on L^2_0 with an eigenfunction of Sobolev regularity.*

The dependence of the lower bound in (2.31) on γ can be intuitively explained as follows. In case of non-weakly-mixing maps the eigenfunctions of U_F are, in general, responsible for slowing down the dissipation. The rate of the dissipation is affected by the regularity of the smoothest eigenfunction. In principle, irregular functions undergo faster dissipation giving rise to slower asymptotics of τ_c. It is not clear, however, whether the actual asymptotics of the dissipation time will be slower than power law in case when all eigenfunctions of U_F on $L^2_0(\mathbb{T}^d)$ are outside any space $H^\gamma(\mathbb{T}^d)$ with $\gamma > 0$.

The above theorem serves as a source of examples of 'non-chaotic' ergodic dynamical systems. A typical example of ergodic but not weakly mixing transformations for which this corollary applies is the family of 'irrational' shifts on $\mathbb{T}^d$ i.e. maps $F\boldsymbol{x} = \boldsymbol{x} + \mathbf{c}$ on $\mathbb{T}^d$, where $\mathbf{c} = (c_1, .., c_d)$ is a constant vector such that the numbers $1, c_1, .., c_d$ are linearly independent over rationals. More general and less trivial examples of ergodic maps giving rise to a slow dissipation time will be discussed in Section 3.2.2 (cf. Remark 3.21).

In Corollary 2.7 we have shown that for any map F and arbitrary small $\delta > 0$, the pseudospectrum $\sigma_\delta(T_\epsilon)$ intersects the unit circle for sufficiently small $\epsilon > 0$. If F is not weakly-mixing, the spectral radius of T_ϵ (that is, the modulus of its largest eigenvalue) is believed to converge to 1 when $\epsilon \to 0$, and the associated eigenstate h_ϵ should converge to a "noiseless eigenstate" h. This "spectral stability" has been discussed for several cases in the continuous-time as well as for discrete-time maps on $\mathbb{T}^2$ [71, 100].

On the opposite, if F is an Anosov map on $\mathbb{T}^2$ (see Section 3.3), the spectrum of T_ϵ does not approach the unit circle, but stays away from it uniformly: $r_{sp}(T_\epsilon)$ is smaller

than some $r_0 < 1$ for any $\epsilon > 0$ [21]. Simultaneously, $\|T_\epsilon\| \to 1$, so we have here a clear manifestation of the *nonnormality* of T_ϵ for such a map. In some cases (see [100] and the linear examples of Section 3.4), the operator T_ϵ is even quasinilpotent, meaning that $r_{sp}(T_\epsilon) = 0$ for all $\epsilon > 0$. For such an Anosov map, the spectral radius of T_ϵ is therefore "unstable" or "discontinuous" in the limit $\epsilon \to 0$, while in the same limit the (radius of its) pseudospectrum $\sigma_\delta(T_\epsilon)$ (for $\delta > 0$ fixed) is "stable".
We end this section by determining the coarse-grained dissipation time for non weakly-mixing maps. We have

Proposition 2.15 *Let F be a measure-preserving map. If F is not weakly-mixing then $\tilde{\tau}_c = \infty$ for small enough $\epsilon > 0$.*

Proof. Let $h \in L^2_0(\mathbb{T}^d)$ be a normalized eigenfunction of U_F, then

$$\begin{aligned} \|\tilde{T}_\epsilon^{(n)} h\| &= \|G_\epsilon U_F^n (h + (G_\epsilon - I)h)\| \geq \|G_\epsilon h\| - \|G_\epsilon U_F^n (G_\epsilon - I)h\| \\ &\geq 1 - 2\|(G_\epsilon - I)h\|. \end{aligned}$$

Since the RHS above is independent of n, we see that $\|\tilde{T}_\epsilon^{(n)}\|$ is close to 1 for all times and sufficiently small $\epsilon > 0$. ■ Thus as opposed to the noisy case (see Prop. 2.10), the coarse-grained evolution through a non-weakly-mixing map does not dissipate.

2.5 Local expansion rate and general lower bound

We saw in the previous section that there exists no general upper bound for coarse-grained dynamics $\tilde{\tau}_c$. On the opposite, we will prove below a general *lower* bound for both coarse-grained and noisy evolutions, valid for any measure-preserving map F of regularity C^1. We note that Propositions 2.8 and 2.15*i)* (which are valid independently of any regularity assumption) do not provide an explicit lower bound.
First we introduce some notation. For any map $F \in C^1$, $DF(\boldsymbol{x})$ is the tangent map of F at the point $\boldsymbol{x} \in \mathbb{T}^d$, mapping a tangent vector at $\boldsymbol{x}$ to a tangent vector at

$F(\boldsymbol{x})$. Selecting the canonical (i.e. Cartesian) basis and metrics on $T(\mathbb{T}^d)$, this map can be represented as a $d \times d$ matrix. The metrics naturally yields a norm $\boldsymbol{v} \in T_{\boldsymbol{x}}(\mathbb{T}^d) \mapsto |v|$ on the tangent space, and therefore a norm on this matrix: $|DF(\boldsymbol{x})| = \max_{|\boldsymbol{v}|=1} |DF(\boldsymbol{x}) \cdot \boldsymbol{v}|$. We are now in position to define the maximal expansion rate of F:

$$\mu_F = \limsup_{n\to\infty} \|DF^n\|_\infty^{1/n}, \quad \text{where} \quad \|DF^n\|_\infty = \sup_{\boldsymbol{x}\in\mathbb{T}^d} |(DF^n)(\boldsymbol{x})|.$$

Since F preserves the Lebesgue measure, the Jacobian $J_F(\boldsymbol{x})$ satisfies $|J_F(\boldsymbol{x})| \geq 1$ at all points. In the Cartesian basis, $J_F(\boldsymbol{x}) = \det(DF(\boldsymbol{x}))$, so that we have $\|DF^n(\boldsymbol{x})\| \geq 1$ for all $\boldsymbol{x} \in \mathbb{T}^d$, $n \geq 0$. One can actually prove the following:

Remark 2.16 *Although $|(DF^n)(\boldsymbol{x})|$ and $\|DF\|_\infty$ may depend on the choice of the metrics, μ_F does not, and satisfies $1 \leq \mu_F \leq \|DF\|_\infty$.*

From the definition of μ_F, for any $\mu > \mu_F$ there exists a constant $A \geq 1$ such that

$$\forall n \in \mathbb{N}, \qquad \|DF^n\|_\infty \leq A\mu^n. \tag{2.32}$$

In some cases one may take $\mu = \mu_F$ in the RHS. In case $\mu_F = 1$, $\|DF^n\|_\infty$ can sometimes grow as a power-law:

$$\|DF^n\|_\infty \leq An^\beta, \qquad n \in \mathbb{N} \tag{2.33}$$

for some $\beta > 0$, or even be uniformly bounded by a constant ($\beta = 0$).

The relationship between, on one side, the local expansion of the map F and on the other side, the dissipation time, can be intuitively understood as follows. A lack of expansion ($\|DF\|_\infty = 1$) results in the transformation of "soft" or "long-wavelength" oscillations into "soft oscillations", both being little affected by the noise operator G_ϵ. On the opposite, a locally strictly expansive map ($\|DF\|_\infty > 1$) will quickly transform soft oscillations into "hard" or "short-wavelength", the latter being much more damped by the noise.

The following theorem precisely measures this relationship, in terms of *lower bounds* for the dissipation times.

Theorem 2.17 *Let F be a measure-preserving C^1 map on $\mathbb{T}^d$, and assume that the noise generating density g satisfies (2.12) or (2.14) for some $\alpha \in (0,2]$.*
i) If $\|DF\|_\infty > 1$, resp. $\mu_F > 1$, then there exist a constant c, resp. constants $\mu \geq \mu_F$ and $\tilde{c}$, such that for small enough ϵ,

$$\tau_c \geq \frac{\alpha \wedge 1}{\ln(\|DF\|_\infty)} \ln(\epsilon^{-1}) + c, \quad \textit{resp.} \quad \tilde{\tau}_c \geq \frac{\alpha \wedge 1}{\ln \mu} \ln(\epsilon^{-1}) + \tilde{c}. \tag{2.34}$$

If F is a C^1 diffeomorphism, then (2.34) holds with $\|DF\|_\infty$ replaced by $\|DF\|_\infty \wedge \|D(F^{-1})\|_\infty$, resp. with some $\mu \geq \mu_F \wedge \mu_{F^{-1}}$.
ii) If $\|DF\|_\infty = 1$ then T_ϵ has slow dissipation time, $\tau_c \gtrsim \epsilon^{-(\alpha\wedge 1)}$. If the noise kernel satisfies the condition (2.14) for $\alpha \in (0,1]$, then the dissipation time is simple, $\tau_c \sim \epsilon^{-\alpha}$.
iii) If $\mu_F = 1$ and $\|DF^n\|_\infty$ grows as a power-law as in Eq. (2.33) with $\beta > 0$, then $\tilde{\tau}_c \gtrsim \epsilon^{-(\alpha\wedge 1)/\beta}$. If $\|DF^n\|_\infty$ is uniformly bounded above by a constant, then $\tilde{\tau}_c = \infty$ for small enough ϵ.

Remark 2.18 *This theorem shows that classical systems on $\mathbb{T}^d$ (i.e. C^1 diffeomorphisms) cannot have a dissipation time growing slower than $C\ln(\epsilon^{-1})$. In view of the results for toral automorphisms (cf. Proposition 4), this lower bound on the dissipation time is sharp and consistent with Kouchnirenko's upper bound on the entropy of the classical systems, namely all classical systems have a finite (possibly zero) Kolmogorov-Sinai entropy (Theorem 12.35. in [10], see also [11], [74]).*

Proof of the Theorem 2.17. The following trivial lemma (similar to Lemma 2.11) will be crucial in the proof.

Lemma 2.19 *Assume that there exists some* $\alpha > 0$ *and a strictly increasing function* $\gamma(n)$, $\gamma(0) = 0$ *such that*

$$\forall n \geq 1, \qquad \|T_\epsilon^n\| \geq 1 - \epsilon^\alpha \gamma(n). \tag{2.35}$$

Then the dissipation time is bounded from below as:

$$\tau_c \geq \gamma^{(-1)}\left(\frac{1-e^{-1}}{\epsilon^\alpha}\right), \tag{2.36}$$

where $\gamma^{(-1)}$ *is the inverse function of* γ.

The same statement holds for the coarse-grained version.

Our task is therefore to bound $\|T_\epsilon^n\|$ (resp. $\|\tilde{T}_\epsilon^{(n)}\|$) from below. A simple computation shows that for any $f \in C^0(\mathbb{T}^d)$, $\|G_\epsilon f\|_\infty \leq \|f\|_\infty$. Since convolution commutes with differentiation, for $f \in C^1$ we also have $\|\nabla(G_\epsilon f)\|_\infty \leq \|\nabla f\|_\infty$. We use this fact to estimate the gradient of $T_\epsilon f$:

$$\begin{aligned}
\|\nabla(T_\epsilon f)\|_\infty &= \|\nabla(G_\epsilon U_F f)\|_\infty \\
&\leq \|\nabla(f \circ F)\|_\infty = \|(\nabla f) \circ F \cdot DF\|_\infty \\
&\leq \|(\nabla f) \circ F\|_\infty \|DF\|_\infty = \|\nabla f\|_\infty \|DF\|_\infty.
\end{aligned}$$

Repeating the above procedure m times, we get

$$\|\nabla(T_\epsilon^m f)\|_\infty \leq \|\nabla f\|_\infty \|DF\|_\infty^m, \quad \|\nabla(U_F T_\epsilon^m f)\|_\infty \leq \|\nabla f\|_\infty \|DF\|_\infty^{m+1}. \tag{2.37}$$

We now choose some arbitrary $f \in C_0^1(\mathbb{T}^d)$, with $\|f\| = 1$. We first apply the triangle inequality:

$$\|T_\epsilon^n f\| = \|G_\epsilon U_F T_\epsilon^{n-1} f\| \geq \|U_F T_\epsilon^{n-1} f\| - \|(G_\epsilon - I) U_F T_\epsilon^{n-1} f\|.$$

To estimate the second term on the RHS we use the bound (2.19) and the estimate (2.37) to obtain

$$\|T_\epsilon^n f\| \geq \|T_\epsilon^{n-1} f\| - C\epsilon^{\alpha \wedge 1} \|\nabla f\|_\infty \|DF\|_\infty^n.$$

Applying the same procedure iteratively to the first term on the RHS, we finally get (remember $\|f\| = 1$):

$$\|T_\epsilon^n\| \geq \|T_\epsilon^n f\| \geq 1 - C\epsilon^{\alpha\wedge 1}\|\nabla f\|_\infty \sum_{m=1}^{n} \|DF\|_\infty^m. \tag{2.38}$$

The computations in the case of the coarse-grained operator are even simpler:

$$\begin{aligned}
\|\tilde{T}_\epsilon^{(n)} f\| &= \|G_\epsilon U_F^n G_\epsilon f\| \\
&\geq 1 - C\epsilon^{\alpha\wedge 1}\|\nabla f\|_\infty - C\epsilon^{\alpha\wedge 1}\|\nabla(G_\epsilon f)\|_\infty \|DF^n\|_\infty \\
&\geq 1 - 2C\epsilon^{\alpha\wedge 1}\|\nabla f\|_\infty \|DF^n\|_\infty.
\end{aligned} \tag{2.39}$$

Notice that from the assumptions on f, $\|\nabla f\|_\infty$ cannot be made arbitrary small, but is necessarily larger than some positive constant. We choose some arbitrary function, say $f = \boldsymbol{e_k}$ with $\boldsymbol{k} = (1, 0)$ which satisfies $\|\nabla f\|_\infty = 2\pi$.

The estimate (2.38) has the form given in Lemma 2.19. The growth of the function $\gamma(n)$ depends on whether $\|DF\|_\infty$ is equal to or larger than 1, which explains why the lower bounds are qualitatively different in the two cases.

In case $\|DF\|_\infty$ is strictly larger than 1, then the function $\gamma(n)$ grows like an exponential, therefore the lower bound is of the type (2.34). For the coarse-grained version, a growth of $\|DF\|_\infty$ of the type (2.32) yields the lower bound for $\tilde{\tau}_c$ in (2.34).

In the case $\|DF\|_\infty = 1$, $\gamma(n)$ is a linear function, so that $\tau_c \geq \frac{1-e^{-1}}{C\|\nabla f\|_\infty}\epsilon^{-(\alpha\wedge 1)}$.

In the coarse-grained version, if $\mu_F = 1$ and $\|DF^n\|_\infty$ grows like in (2.33) with $\beta > 0$, the dissipation is slow: $\tilde{\tau}_c \geq C\epsilon^{-(\alpha\wedge 1)/\beta}$. In the case where $\|DF^n\|_\infty$ is uniformly bounded by some constant, the norm of the coarse-grained propagator stays larger than some positive constant for all times, so that for small enough noise $\tilde{\tau}_c$ is infinite. ■

2.6 Decay of correlations and general upper bound

For any two functions $f, g \in L^2_0(\mathbb{T}^d)$ the dynamical correlation function for the map F is defined as the following function of $n \in \mathbb{N}$ (see e.g. [12]):

$$C_{f,g}(n) = C^0_{f,g}(n) = m(fU^n_F g) = \langle \bar{f}, U^n_F g\rangle = \langle P^n_F \bar{f}, g\rangle.$$

The same quantity may be defined for the noisy evolution:

$$C^\epsilon_{f,g}(n) = m(fT^n_\epsilon g).$$

We recall that a map F is *mixing* iff for any $f,\ g \in L^2_0$,

$$C_{f,g}(n) \to 0, \quad \text{as} \quad n \to \infty.$$

The correlation function can easily be measured in (numerical or real-life) experiments, so it is often used to characterize the dynamics of a system.

To focus the attention, we will only be concerned with maps for which correlations decay in a precise way. We assume that there exist Hölder exponents $s_*, s \in \mathbb{R}_+$, $0 \le s_* \le s$ together with some decreasing function $\Gamma(n) = \Gamma_{s_*,s}(n)$ with $\Gamma(n) \xrightarrow{n\to\infty} 0$, such that for any observables $f \in C^{s_*}_0(\mathbb{T}^d)$, $g \in C^s_0(\mathbb{T}^d)$ and for sufficiently small $\epsilon \ge 0$ (sometimes only for $\epsilon = 0$),

$$\forall n \in \mathbb{N}, \quad |C^\epsilon_{f,g}(n)| \le \|f\|_{C^{s_*}} \|g\|_{C^s} \Gamma(n). \tag{2.40}$$

In general, such a bound can be proved only if the map F has regularity C^{s+1}. The reason why we do not necessarily take the same norm for the functions f and g will be clear below.

We will be mainly interested in the following two types of decay

i) Power-law decay: there exists $C > 0$, $\beta > 0$ such that,

$$\Gamma(n) = Cn^{-\beta}. \tag{2.41}$$

This behavior is characteristic of intermittent maps, e.g. maps possessing one or several neutral orbits [13].

ii) Exponential decay: there exists $C > 0$, $0 < \sigma < 1$ such that,

$$\Gamma(n) = C\sigma^n. \tag{2.42}$$

Such a behavior was proved in the case of uniformly expanding or hyperbolic maps on the torus (see Section 3.4), as well as many other cases [13].

The central result of this section is a relationship between, on one side, the decay of noisy (resp. noiseless) correlations and on the other side, the small-noise behavior of the noisy (resp. coarse-graining) dissipation time. The intuitive picture is similar to the one linking the local expansion rate to the dissipation: namely, a fast decay of correlations is generally due to the transition of "soft" into "hard" fluctuations of the observable through the evolution, which is itself induced by large expansion rates of the map. Still, as opposed to what we obtained in last Section, the following theorem and its corollary yields *upper bounds* for the dissipation time.

Theorem 2.20 *Let F be a volume preserving map on $\mathbb{T}^d$ with correlations decaying as in Eq. (2.40) for some indices s, s_* and decreasing function $\Gamma(n)$, at least in the noiseless limit $\epsilon = 0$. Assume that the noise generating function g is $([s]+1)$-differentiable, and that all its derivatives of order $|\boldsymbol{\alpha}|_1 \leq [s]+1$ satisfy*

$$|D^{\boldsymbol{\alpha}} g(\boldsymbol{x})| \lesssim \frac{1}{|\boldsymbol{x}|^M}, \quad |\boldsymbol{x}| \gg 1,$$

with a power $M > d$.
Then there exist constants $\tilde{C} > 0$, $\epsilon_o > 0$ such that the coarse-grained propagator satisfies

$$\forall \epsilon \leq \epsilon_o, \quad \forall n \geq 0, \qquad \|\tilde{T}_\epsilon^{(n)}\| \leq \tilde{C}\,\frac{\Gamma(n)}{\epsilon^{d+s+s_*}}. \tag{2.43}$$

If the decay of correlations (2.40) *also holds for sufficiently small* $\epsilon > 0$ *(and assuming the Perron-Frobenius operator* P_F *is bounded in* $C^s(\mathbb{T}^d)$*), then the noisy operator satisfies (for some constants* $C > 0$, $\epsilon_o > 0$*):*

$$\forall \epsilon \leq \epsilon_o, \quad \forall n \geq 0, \qquad \|T_\epsilon^n\| \leq C\, \frac{\Gamma(n)}{\epsilon^{d+s+s_*}}. \tag{2.44}$$

From these estimates, we straightforwardly obtain the following bounds on both dissipation times (the assumptions on F and the noise generating function g are the same as in the Theorem):

Corollary 2.21 *I) If the correlation function satisfies the bound* (2.40) *for* $\epsilon = 0$*, then the coarse-grained dissipation time is well defined* ($\tilde{\tau}_c < \infty$)*. Moreover,*

i) if $\Gamma(n) \sim n^{-\beta}$ *then there exists a constant* $\tilde{C} > 0$ *such that*

$$\tilde{\tau}_c \leq \tilde{C} \epsilon^{-\frac{d+s+s_*}{\beta}}$$

ii) if $\Gamma(n) \sim \sigma^n$ *then there exists a constant* $\tilde{c}$ *such that*

$$\tilde{\tau}_c \leq \frac{d+s+s_*}{|\ln \sigma|} \ln(\epsilon^{-1}) + \tilde{c},$$

II) If Eq. (2.40) *holds for sufficiently small* $\epsilon > 0$*, then*

i) if $\Gamma(n) \sim n^{-\beta}$*, there exists a constant* $C > 0$ *such that*

$$\tau_c \leq C \epsilon^{-\frac{d+s+s_*}{\beta}}$$

ii) if $\Gamma(n) \sim \sigma^n$*, there exists a constant* c *such that*

$$\tau_c \leq \frac{d+s+s_*}{|\ln \sigma|} \ln(\epsilon^{-1}) + c.$$

Proof of Theorem 2.20.

<u>1^{st} step</u>: We represent the action of T_ϵ^n (resp. $\tilde{T}_\epsilon^{(n)}$) on an observable $f \in L_0^2(\mathbb{T}^d)$ in terms of the correlation functions $C^\epsilon(n)$ (resp. $C(n)$). To do this we Fourier

decompose both $T_\epsilon^{n+2}f$ and $f_1 = U_F f$, and use Eq. (2.6):

$$\begin{aligned}
T_\epsilon^{n+2}f &= \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d} \langle \boldsymbol{e_j}, G_\epsilon U_F T_\epsilon^n G_\epsilon f_1\rangle \boldsymbol{e_j} \\
&= \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d}\sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^n} \hat{f}_1(\boldsymbol{k})\langle G_\epsilon \boldsymbol{e_j}, U_F T_\epsilon^n G_\epsilon \boldsymbol{e_k}\rangle \boldsymbol{e_j} \\
&= \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d}\sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \hat{f}_1(\boldsymbol{k})\hat{g}_\epsilon(\boldsymbol{j})\hat{g}_\epsilon(\boldsymbol{k})\langle P_F \boldsymbol{e_j}, T_\epsilon^n \boldsymbol{e_k}\rangle \boldsymbol{e_j}.
\end{aligned}$$

(remember that $\hat{g}$ is a real function). A similar computation for the coarse-grained propagator yields:

$$\tilde{T}_\epsilon^{(n)}f = \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d}\sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{k})\hat{g}_\epsilon(\boldsymbol{j})\hat{g}_\epsilon(\boldsymbol{k})\langle \boldsymbol{e_j}, U_F^n \boldsymbol{e_k}\rangle \boldsymbol{e_j}.$$

Taking the norms on both sides, we get in the noisy case:

$$\begin{aligned}
\|T_\epsilon^{n+2}f\|^2 &= \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^2}\left|\sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \hat{f}_1(\boldsymbol{k})\langle P_F\boldsymbol{e_j}, T_\epsilon^n \boldsymbol{e_k}\rangle \hat{g}_\epsilon(\boldsymbol{j})\hat{g}_\epsilon(\boldsymbol{k})\right|^2 \\
&\leq \sum_{0\neq \boldsymbol{j}\in\mathbb{Z}^d}\left(\sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d}|\hat{f}_1(\boldsymbol{k})|^2\right)\sum_{0\neq \boldsymbol{k}\in\mathbb{Z}^d}|\langle P_F\boldsymbol{e_j}, T_\epsilon^n\boldsymbol{e_k}\rangle|^2|\hat{g}_\epsilon(\boldsymbol{j})\hat{g}_\epsilon(\boldsymbol{k})|^2 \\
\Longrightarrow \|T_\epsilon^{n+2}f\|^2 &\leq \|f_1\|^2\sum_{0\neq \boldsymbol{j},\boldsymbol{k}\in\mathbb{Z}^d}|C^\epsilon_{P_F\boldsymbol{e_{-j}},\boldsymbol{e_k}}(n)|^2|\hat{g}(\epsilon\boldsymbol{j})\hat{g}(\epsilon\boldsymbol{k})|^2, \qquad (2.45)
\end{aligned}$$

and in the coarse-graining case

$$\|\tilde{T}_\epsilon^{(n)}f\|^2 \leq \|f\|^2\sum_{0\neq \boldsymbol{j},\boldsymbol{k}\in\mathbb{Z}^d}|C_{\boldsymbol{e_{-j}},\boldsymbol{e_k}}(n)|^2|\hat{g}(\epsilon\boldsymbol{j})\hat{g}(\epsilon\boldsymbol{k})|^2. \qquad (2.46)$$

These two expressions explicitly relate the dissipation with the correlation functions.

<u>2^{nd} step:</u> We now apply the estimates (2.40) on correlations for the observables $\boldsymbol{e_k}$, $\boldsymbol{e_{-j}}$, $P_F\boldsymbol{e_{-j}}$. In the coarse-grained case, it yields (using simple bounds of the type of Eq. (2.57)):

$$\forall \boldsymbol{j},\ \boldsymbol{k}\in\mathbb{Z}^d\setminus\{0\}, \qquad |C_{\boldsymbol{e_{-j}},\boldsymbol{e_k}}(n)| \leq C'\ |\boldsymbol{j}|^s|\boldsymbol{k}|^{s_*}\Gamma(n).$$

In the noisy case, we need to assume that the Perron-Frobenius operator P_F is bounded in the space $C^s(\mathbb{T}^d)$. This property is in general a prerequisite in the proof of estimates of the type (2.40), so this assumption is quite natural here.

$$\begin{aligned}\forall \boldsymbol{j},\ \boldsymbol{k} \in \mathbb{Z}^d \setminus \{0\}, \qquad |C^{\epsilon}_{P_F \boldsymbol{e}_{-\boldsymbol{j}}, \boldsymbol{e}_{\boldsymbol{k}}}(n)| &\leq C\ \|P_F \boldsymbol{e}_{-\boldsymbol{j}}\|_{C^s} \|\boldsymbol{e}_{\boldsymbol{k}}\|_{C^{s_*}} \Gamma(n) \\ &\leq C \|P_F\|_{C^s} |\boldsymbol{j}|^s |\boldsymbol{k}|^{s_*} \Gamma(n). \end{aligned} \tag{2.47}$$

We insert these bounds on the decay of correlations in the expressions (2.45-2.46), for instance in the coarse-grained case we get:

$$\forall n \geq 0, \qquad \|\tilde{T}_{\epsilon}^{(n)}\|^2 \leq C\ \Gamma(n)^2 \left(\epsilon^{-(s+s_*)} \sum_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} |\epsilon \boldsymbol{k}|^{s+s_*} \hat{g}(\epsilon \boldsymbol{k})^2 \right)^2. \tag{2.48}$$

3^{rd} step: We finally estimate the ϵ-dependence of the RHS of the above inequality. Up to a factor ϵ^{-d}, the sum in the brackets is a Riemann sum for the integral $\int |\boldsymbol{\xi}|^{s+s_*} \hat{g}(\boldsymbol{\xi})^2 d\boldsymbol{\xi} < \infty$. A precise connection is given in the following lemma, proved in Section 2.8:

Lemma 2.22 *Let $f \in C^0(\mathbb{R}^d)$ be symmetric w.r.t. the origin and decaying at infinity as $|f(\boldsymbol{x})| \lesssim |\boldsymbol{x}|^{-M}$ with $M > d$. Then the following small-ϵ estimate holds in the limit $\epsilon \to 0$:*

$$\epsilon^d \sum_{\boldsymbol{k} \in \mathbb{Z}^d} \hat{f}(\epsilon \boldsymbol{k})^2 = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi})^2 d\boldsymbol{\xi} + \mathcal{O}(\epsilon^M). \tag{2.49}$$

Let $m \in \mathbb{N}$ satisfy $2m \leq s + s_* \leq 2m+2$ (notice that $m \leq [s]$ since we assumed $s_* \leq s$). From the obvious inequality

$$\forall x > 0, \qquad x^{s+s_*} \leq x^{2m} + x^{2m+2},$$

we may replace in the RHS of (2.48) the factor $|\epsilon \boldsymbol{k}|^{s+s_*}$ by $|\epsilon \boldsymbol{k}|^{2m} + |\epsilon \boldsymbol{k}|^{2m+2}$. Applying Lemma 2.22 to the derivatives of g of order m and $m+1$, we end up with the following

upper bound, which proves the first part of the theorem:

$$\begin{aligned}\|\tilde{T}_\epsilon^{(n)}\|^2 &\leq C\,\Gamma(n)^2\left(\frac{1}{\epsilon^{d+s+s_*}}\int_{\mathbb{R}^d}\left(|\boldsymbol{\xi}|^{2m}+|\boldsymbol{\xi}|^{2(m+1)}\right)\hat{g}(\boldsymbol{\xi})^2 d\boldsymbol{\xi}+\mathcal{O}(\epsilon^M)\right)^2\\ &\leq C'\,\frac{\Gamma(n)^2}{\epsilon^{2(d+s+s_*)}}\,\|g\|^4_{H^{m+1}}.\end{aligned}$$

The computations follow identically for the case of the noisy operator, yielding the second part of the theorem. ■

2.7 Dissipation time and optimization problems

In general the problem of computing the dissipation time is rather complicated. In some cases it can be reformulated as an asymptotic optimization problem. To see it, one can represent the action of a given unitary operator U in the Fourier basis

$$U\boldsymbol{e}_{\boldsymbol{k}} = \sum_{0\neq \boldsymbol{k}'\in\mathbb{Z}^d} u_{\boldsymbol{k},\boldsymbol{k}'}\boldsymbol{e}_{\boldsymbol{k}'}, \tag{2.50}$$

where for each $\boldsymbol{k}$

$$\sum_{0\neq \boldsymbol{k}'\in\mathbb{Z}^d} |u_{\boldsymbol{k},\boldsymbol{k}'}|^2 = 1. \tag{2.51}$$

Next we introduce the notation

$$\begin{aligned}\mathcal{U}_n(\boldsymbol{k}_0,\boldsymbol{k}_n) &= \sum_{0\neq \boldsymbol{k}_1,\ldots,\boldsymbol{k}_{n-1}\in\mathbb{Z}^d} u_{\boldsymbol{k}_0,\boldsymbol{k}_1}\ldots u_{\boldsymbol{k}_{n-1},\boldsymbol{k}_n}\prod_{l=1}^n \hat{g}_\epsilon(\boldsymbol{k}_l)\\ \mathcal{S}_n(\boldsymbol{k}_n) &= \{\boldsymbol{k}_0\in\mathbb{Z}^d\backslash\{0\} : \mathcal{U}_n(\boldsymbol{k}_0,\boldsymbol{k}_n)\neq 0\}.\end{aligned}$$

Then for any $f\in L_0^2(\mathbb{T}^d)$ we have

$$\begin{aligned}\|T_\epsilon^n f\|^2 &= \left\|\sum_{0\neq\boldsymbol{k}_0\in\mathbb{Z}^d}\hat{f}(\boldsymbol{k}_0)T_\epsilon^n\boldsymbol{e}_{\boldsymbol{k}_0}\right\|^2 = \left\|\sum_{0\neq\boldsymbol{k}_0\in\mathbb{Z}^d}\hat{f}(\boldsymbol{k}_0)\sum_{0\neq\boldsymbol{k}_n\in\mathbb{Z}^d}\mathcal{U}_n(\boldsymbol{k}_0,\boldsymbol{k}_n)\boldsymbol{e}_{\boldsymbol{k}_n}\right\|^2\\ &= \sum_{0\neq\boldsymbol{k}_n\in\mathbb{Z}^d}\left|\sum_{0\neq\boldsymbol{k}_0\in\mathbb{Z}^d}\hat{f}(\boldsymbol{k}_0)\mathcal{U}_n(\boldsymbol{k}_0,\boldsymbol{k}_n)\right|^2 \qquad (2.52)\\ &= \sum_{0\neq\boldsymbol{k}_n\in\mathbb{Z}^d}\left|\sum_{\boldsymbol{k}_0\in\mathcal{S}_n(\boldsymbol{k}_n)}\hat{f}(\boldsymbol{k}_0)\mathcal{U}_n(\boldsymbol{k}_0,\boldsymbol{k}_n)\right|^2. \qquad (2.53)\end{aligned}$$

The following general upper bound for $\|T_\epsilon^n f\|$ holds.

Lemma 2.23 *For any $f \in L_0^2(\mathbb{T}^d)$,*

$$\|T_{\epsilon,\alpha}^n f\|^2 \;\leq\; \sum_{0\neq \boldsymbol{k}_n \in \mathbb{Z}^d} \sum_{\boldsymbol{k}_0 \in \mathcal{S}_n(\boldsymbol{k}_n)} |\hat{f}(\boldsymbol{k}_0)|^2 \sum_{\boldsymbol{k}_0 \in \mathcal{S}_n(\boldsymbol{k}_n)} |\mathcal{U}_n(\boldsymbol{k}_0, \boldsymbol{k}_n)|^2. \tag{2.54}$$

For the proof we refer to Section 2.8.

In order to see how this lemma can work in practice let us consider a concrete example. To this end we focus on a case when $u_{\boldsymbol{k},\boldsymbol{k}'}$ is a Kronecker's delta function

$$u_{\boldsymbol{k},\boldsymbol{k}'} = \delta_{A\boldsymbol{k},\boldsymbol{k}'}, \tag{2.55}$$

where $A : \mathbb{Z}^d \mapsto \mathbb{Z}^d$ is a linear surjective map.

Under this assumption the upper bound (2.54) can be used to obtain an identity for $\|T_\epsilon^n\|$. Indeed, first observe that

$$\mathcal{U}_n(\boldsymbol{k}_0, \boldsymbol{k}_n) = \prod_{l=1}^{n} \hat{g}_\epsilon(A^l \boldsymbol{k}_0)\delta_{A^n \boldsymbol{k}_0, \boldsymbol{k}_n}$$

and hence (2.54) becomes

$$\|T_\epsilon^n f\|^2 \;\leq\; \sum_{0\neq \boldsymbol{k}_0 \in \mathbb{Z}^d} |\hat{f}(\boldsymbol{k}_0)|^2 \prod_{l=1}^{n} \hat{g}_\epsilon(A^l \boldsymbol{k}_0) \leq \|f\|^2 \max_{0\neq \boldsymbol{k} \in \mathbb{Z}^d} \prod_{l=1}^{n} \hat{g}_\epsilon(A^l \boldsymbol{k})$$

On the other hand for any nonzero $\boldsymbol{k} \in \mathbb{Z}^d$, one can take in (2.52) $f = \boldsymbol{e}_{\boldsymbol{k}}$ and get

$$\|T_\epsilon^n f\|^2 = \prod_{l=1}^{n} \hat{g}_\epsilon(A^l \boldsymbol{k})$$

and therefore

$$\|T_\epsilon^n\| \;=\; \max_{0\neq \boldsymbol{k} \in \mathbb{Z}^d} \prod_{l=1}^{n} \hat{g}_\epsilon(A^l \boldsymbol{k}). \tag{2.56}$$

Let us now determine the class of maps F such that the corresponding Koopman operator U_F satisfies (2.55). The relation (2.55) implies

$$U_F \boldsymbol{e}_{\boldsymbol{k}} = \boldsymbol{e}_{A\boldsymbol{k}} = e^{2\pi i \langle A\boldsymbol{k}, \boldsymbol{x}\rangle}.$$

On the other hand

$$U_F \boldsymbol{e}_{\boldsymbol{k}}(\boldsymbol{x}) = \boldsymbol{e}_{\boldsymbol{k}}(F\boldsymbol{x}) = e^{2\pi i \langle \boldsymbol{k}, F\boldsymbol{x}\rangle}.$$

Thus

$$\langle \boldsymbol{k}, F\boldsymbol{x}\rangle = \langle A\boldsymbol{k}, \boldsymbol{x}\rangle \text{ mod } 1, \quad \forall \boldsymbol{x} \in \mathbb{R}^d,\ \boldsymbol{k} \in \mathbb{Z}^d,$$

that is, A is linear and $A^\dagger$ equals the lifting of F from $\mathbb{T}^d$ onto $\mathbb{R}^d$. Moreover, the matrix A has integer entries and determinant equal to ± 1, i.e., A (and F) is a toral automorphism. In the next Chapter we will use formula 2.56 to derive an exact asymptotics of the dissipation time for virtually all toral automorphisms.

2.8 Technical proofs

Proof of Lemma 2.1

We use the following upper bound: for any $\alpha \in (0, 2]$, there is a constant C_α such that

$$\forall x \in \mathbb{R}, \quad 0 \le 1 - \cos(2\pi x) \le C_\alpha |x|^\alpha. \tag{2.57}$$

Besides, one has the asymptotics $1 - \cos(x) \approx \frac{x^2}{2}$ for small x. We simply apply these estimates to the following integral:

$$\begin{aligned} 1 - \hat{g}(\boldsymbol{\xi}) &= \int_{\mathbb{R}^d} (1 - \cos(2\pi \boldsymbol{x} \cdot \boldsymbol{\xi})) g(\boldsymbol{x}) d\boldsymbol{x} \\ &\le \int_{\mathbb{R}^d} C_\alpha |\boldsymbol{x} \cdot \boldsymbol{\xi}|^\alpha g(\boldsymbol{x}) d\boldsymbol{x} \\ &\le C_\alpha |\boldsymbol{\xi}|^\alpha \int_{\mathbb{R}^d} |\boldsymbol{x}|^\alpha g(\boldsymbol{x}) d\boldsymbol{x} = C_\alpha M_\alpha |\boldsymbol{\xi}|^\alpha. \end{aligned}$$

In the case g admits a second moment, we have in the limit $\boldsymbol{\xi} \to 0$:

$$\begin{aligned} \int_{\mathbb{R}^d} (1 - \cos(2\pi \boldsymbol{x} \cdot \boldsymbol{\xi})) g(\boldsymbol{x}) d\boldsymbol{x} &\approx \int_{\mathbb{R}^d} 2\pi^2 (\boldsymbol{x} \cdot \boldsymbol{\xi})^2 g(\boldsymbol{x}) d\boldsymbol{x} \\ &\approx 2\pi^2 |\boldsymbol{\xi}|^2 \int_{\mathbb{R}^d} (\boldsymbol{x} \cdot \hat{\boldsymbol{\xi}})^2 g(\boldsymbol{x}) d\boldsymbol{x}, \end{aligned}$$

where we have used the notation $\hat{\boldsymbol{\xi}} = \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}$ for any $\boldsymbol{\xi} \neq 0$. ■

Proof of Proposition 2.2

The statement i) is standard in the context of distributions [125, p.157]. In our case, assume that $f \in L^2$ is normalized to unity and consider an arbitrary small $\delta > 0$. Since $f \in L^2(\mathbb{T}^d)$, there exists $K > 0$ s.t. $\sum_{|\boldsymbol{k}|\geq K} |\hat{f}(\boldsymbol{k})|^2 < \delta$. Since $\hat{g}$ is continuous and $\hat{g}(0) = 1$, there exists η such that $(1 - \hat{g}(\boldsymbol{\xi}))^2 < \delta$ if $|\boldsymbol{\xi}| < \eta$. Thus using spectral decomposition (2.6) of G_ϵ, we obtain for all $\epsilon < \frac{\eta}{K}$

$$\|G_\epsilon f - f\|^2 = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (1 - \hat{g}(\epsilon\boldsymbol{k}))^2 |\hat{f}(\boldsymbol{k})|^2 \leq \delta \sum_{|\boldsymbol{k}|<K} |\hat{f}(\boldsymbol{k})|^2 + \sum_{|\boldsymbol{k}|>K} |\hat{f}(\boldsymbol{k})|^2 \leq 2\delta. \quad (2.58)$$

To prove the next statement, first notice that if g satisfies the estimate (2.13) for the exponent α, it also satisfies it for the exponent $\gamma \wedge \alpha$. Using once again spectral decomposition of G_ϵ, and applying the estimate (2.13) with the latter exponent we get

$$\begin{aligned}\|G_\epsilon f - f\|^2 &\leq \sum_{\boldsymbol{k}\in\mathbb{Z}^d} (C_{\gamma\wedge\alpha} M_{\gamma\wedge\alpha} |\epsilon \boldsymbol{k}|^{\gamma\wedge\alpha})^2 |\hat{f}(\boldsymbol{k})|^2 \\ &\leq (C_{\gamma\wedge\alpha} M_{\gamma\wedge\alpha})^2 \epsilon^{2(\gamma\wedge\alpha)} \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\boldsymbol{k}|^{2(\gamma\wedge\alpha)} |\hat{f}(\boldsymbol{k})|^2 \\ &\leq (C_{\gamma\wedge\alpha} M_{\gamma\wedge\alpha})^2 \epsilon^{2(\gamma\wedge\alpha)} \|f\|^2_{H^{\gamma\wedge\alpha}}.\end{aligned} \quad (2.59)$$

To obtain the last statement, we notice that any $f \in C^1(\mathbb{T}^d)$ is automatically in $H^1(\mathbb{T}^d)$, and that its gradient satisfies

$$\|\nabla f\|^2_\infty \geq \|\nabla f\|^2 = 4\pi^2 \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\boldsymbol{k}|^2 |\hat{f}(\boldsymbol{k})|^2 \geq 4\pi^2 \sum_{\boldsymbol{k}\in\mathbb{Z}^d} |\boldsymbol{k}|^{2(1\wedge\alpha)} |\hat{f}(\boldsymbol{k})|^2.$$

The inequality (2.59) with $\gamma = 1$ then yields the desired result. ■

Proof of Proposition 2.5

The proof will be based on the Riesz convexity theorem (see [131], pp. 93-100) which states that for any operator T defined on $L^p(\mathbb{T}^d), 1 \leq p \leq \infty$, $\ln \|T\|_p$ is a convex function of p^{-1}. On the space $L^p(\mathbb{T}^d)$ we consider the operator $\widetilde{T} := T_{\epsilon,\alpha} - \langle\cdot\rangle$ and we

have the relation $\widetilde{T}^n f = T^n_{\epsilon,\alpha}(f - \langle f \rangle), \forall f \in L^p(\mathbb{T}^d), n \geq 1$ because $T_{\epsilon,\alpha}$ is conservative. Now since $\|f - \langle f \rangle\|_p \leq 2\|f\|_p$, it follows that

$$\|\widetilde{T}^n\|_p \leq 2\|T^n_{\epsilon,\alpha}\|_{p,0} \leq 2 \tag{2.60}$$

$$\|T^n_{\epsilon,\alpha}\|_{p,0} \leq \|\widetilde{T}^n\|_p \tag{2.61}$$

for $1 \leq p \leq \infty, n \geq 1$. The Riesz convexity theorem implies that if $p < q < \infty$

$$\ln \|\widetilde{T}^n\|_q \leq \frac{p}{q} \ln \|\widetilde{T}^n\|_p + \left(1 - \frac{p}{q}\right) \ln \|\widetilde{T}^n\|_\infty \tag{2.62}$$

while if $1 < q < p$

$$\ln \|\widetilde{T}^n\|_q \leq \left(\frac{1 - 1/q}{1 - 1/p}\right) \ln \|\widetilde{T}^n\|_p + \left(1 - \frac{1 - 1/q}{1 - 1/p}\right) \ln \|\widetilde{T}^n\|_1. \tag{2.63}$$

From (2.62)-(2.63) we have the interpolation relations

$$\|\widetilde{T}^n\|_q \leq \|\widetilde{T}^n\|_p^{p/q} \|\widetilde{T}^n\|_\infty^{1-p/q}, \quad p < q < \infty \tag{2.64}$$

$$\|\widetilde{T}^n\|_q \leq \|\widetilde{T}^n\|_p^{(1-q^{-1})/(1-p^{-1})} \|\widetilde{T}^n\|_1^{1-(1-q^{-1})/(1-p^{-1})}, \quad 1 < q < p \tag{2.65}$$

which, along with (2.60)-(2.61), imply

$$\|T^n_{\epsilon,\alpha}\|_{q,0} \leq 2\|T^n_{\epsilon,\alpha}\|_{p,0}^{p/q}, \quad p < q < \infty$$

$$\|T^n_{\epsilon,\alpha}\|_{q,0} \leq 2\|T^n_{\epsilon,\alpha}\|_{p,0}^{(1-q^{-1})/(1-p^{-1})}, \quad 1 < q < p$$

This proves that the order of divergence of $n_{diss}(p)$ are the same for $1 < p < \infty$. Estimates (2.64)-(2.65) also show that the order of divergence of $n_{diss}(1)$ and $n_{diss}(\infty)$ is at least as high as $n_{diss}(p), 1 < p < \infty$. ■

Proof of Proposition 2.7

We prove the limit $d_\epsilon(1) \xrightarrow{\epsilon \to 0} 0$ by contradiction. Assume that there is some constant $a \in (0, 1)$ such that for all $\epsilon > 0$, the distance $d_\epsilon(1) > a$. We will show that the following triangle inequality holds:

$$\forall \epsilon > 0, \quad d_\epsilon(1 - a/2) > a/2. \tag{2.66}$$

First of all, notice that the assumption $d_\epsilon(1) > a$ means that for any $\lambda \in S^1$, $\|R_\epsilon(\lambda)\| < a^{-1}$. We apply the following identity [125]:

$$R_\epsilon(\lambda') = R_\epsilon(\lambda)\{1 + \sum_{n\geq 1}(\lambda - \lambda')^n R_\epsilon(\lambda)^n\}$$

with $\lambda' = r\lambda$, for $1 - a < r < 1$. Taking norm of both sides yields the bound $\|R_\epsilon(\lambda')\| \leq \frac{1}{r-(1-a)}$, uniformly w.r.t ϵ. Since this upper bound holds for any $|\lambda'| = r$, it shows that the spectral radius $r_{sp}(T_\epsilon) \leq 1-a$, and proves (2.66) by taking $r = 1-a/2$. We can now use (2.66) in the upper bound (2.29) of Theorem 2.9: this ϵ-independent upper bound shows that τ_* remains finite in the limit $\epsilon \to 0$, which contradicts Proposition 2.8. ∎

Proof of Lemma 2.22

Considering its decay at infinity, the function f is automatically in $L^2(\mathbb{R}^d)$. The function $\hat{f}^2$ is the Fourier transform of the self-convolution $f * f$. Therefore, using the parity of f and applying the Poisson summation formula to the LHS of (2.49) yields

$$\epsilon^d \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \hat{f}(\epsilon\boldsymbol{k})^2 = \int \hat{f}^2(\boldsymbol{\xi})d\boldsymbol{\xi} + \sum_{0\neq\boldsymbol{n}\in\mathbb{Z}^d} (f * f)\Big(\frac{\boldsymbol{n}}{\epsilon}\Big). \tag{2.67}$$

A simple computation shows that $(f * f)(\boldsymbol{x})$ also decays as fast as $|\boldsymbol{x}|^{-M}$. This piece of information is now sufficient to control the RHS of (2.67), yielding the result, Eq. (2.49). ∎

Proof of Lemma 2.23

Using the notation introduced in Section 2.7 one has

$$T^n_{\epsilon,\alpha}\boldsymbol{f}_{\boldsymbol{k}_0} = (G_{\epsilon,\alpha}U)^n\boldsymbol{f}_{\boldsymbol{k}_0} = (G_{\epsilon,\alpha}U)^{n-1}\sum_{0\neq\boldsymbol{k}_1\in\mathbb{Z}^d} u_{\boldsymbol{k}_0,\boldsymbol{k}_1}e^{-\epsilon|\boldsymbol{k}_1|^{2\alpha}}\boldsymbol{f}_{\boldsymbol{k}_1}$$

$$= \sum_{0\neq\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n\in\mathbb{Z}^d} u_{\boldsymbol{k}_0,\boldsymbol{k}_1}u_{\boldsymbol{k}_1,\boldsymbol{k}_2}\ldots u_{\boldsymbol{k}_{n-1},\boldsymbol{k}_n}e^{-\epsilon\sum_{l=1}^n|\boldsymbol{k}_l|^{2\alpha}}\boldsymbol{f}_{\boldsymbol{k}_n} = \sum_{0\neq\boldsymbol{k}_n\in\mathbb{Z}^d}\mathcal{U}_n(\boldsymbol{k}_0,\boldsymbol{k}_n)\boldsymbol{f}_{\boldsymbol{k}_n}.$$

We note that for any n and $\boldsymbol{k}_n \in \mathbb{Z}^d$, the sequence $\mathcal{U}_n(\boldsymbol{k}_0, \boldsymbol{k}_n)$ (indexed by $\boldsymbol{k}_0 \in \mathbb{Z}^d$) belongs to $l^2(\mathbb{Z}^d)$. Indeed, using the Cauchy-Schwarz inequality and identity (2.51) one gets for $n = 2$,

$$
\begin{aligned}
\sum_{0\neq \boldsymbol{k}_0\in\mathbb{Z}^d} |\mathcal{U}_2(\boldsymbol{k}_0,\boldsymbol{k}_2)|^2 &= \sum_{0\neq \boldsymbol{k}_0\in\mathbb{Z}^d} \left| \sum_{0\neq \boldsymbol{k}_1\in\mathbb{Z}^d} u_{\boldsymbol{k}_0,\boldsymbol{k}_1} u_{\boldsymbol{k}_1,\boldsymbol{k}_2} e^{-\epsilon(|\boldsymbol{k}_1|^{2\alpha}+|\boldsymbol{k}_2|^{2\alpha})} \right|^2 \\
\leq \quad & \sum_{0\neq \boldsymbol{k}_0\in\mathbb{Z}^d} \sum_{0\neq \boldsymbol{k}_1\in\mathbb{Z}^d} |u_{\boldsymbol{k}_0,\boldsymbol{k}_1}|^2 e^{-\epsilon|\boldsymbol{k}_1|^{2\alpha}} \sum_{0\neq \boldsymbol{k}_1\in\mathbb{Z}^d} |u_{\boldsymbol{k}_1,\boldsymbol{k}_2}|^2 e^{-\epsilon|\boldsymbol{k}_1|^{2\alpha}} e^{-2\epsilon|\boldsymbol{k}_2|^{2\alpha}} \\
\leq \quad & \sum_{0\neq \boldsymbol{k}_1\in\mathbb{Z}^d} e^{-\epsilon|\boldsymbol{k}_1|^{2\alpha}} \sum_{0\neq \boldsymbol{k}_1\in\mathbb{Z}^d} e^{-\epsilon|\boldsymbol{k}_1|^{2\alpha}} e^{-2\epsilon|\boldsymbol{k}_2|^{2\alpha}} = K e^{-2\epsilon|\boldsymbol{k}_2|^{2\alpha}},
\end{aligned}
$$

where K denotes a constant which depends only on ϵ and α. Similar estimates hold for $n > 2$.

Now applying the Cauchy-Schwarz inequality in (2.52) we get

$$
\|T^n_{\epsilon,\alpha} f\|^2 \;\leq\; \sum_{0\neq \boldsymbol{k}_n\in\mathbb{Z}^d} \sum_{\boldsymbol{k}_0\in\mathcal{S}_n(\boldsymbol{k}_n)} |\hat{f}(\boldsymbol{k}_0)|^2 \sum_{\boldsymbol{k}_0\in\mathcal{S}_n(\boldsymbol{k}_n)} |\mathcal{U}_n(\boldsymbol{k}_0,\boldsymbol{k}_n)|^2. \tag{2.68}
$$

■

Chapter 3

Dissipation time of classically chaotic systems

3.1 Dissipation time of toral automorphisms

3.1.1 Preliminaries

It is well known (see [1]) that (the lifting map corresponding to) any toral homeomorphism $H : \mathbb{T}^d \mapsto \mathbb{T}^d$ can be decomposed into three parts $H = L + P + c$, where L, the linear part, is an element of $SL(d, \mathbb{Z})$ - the set of all matrices with integer entries and determinant equal to ± 1, P is periodic i.e. $P(\boldsymbol{x} + \boldsymbol{v}) = P(\boldsymbol{x})$ for any $\boldsymbol{v} \in \mathbb{Z}^d$, and c is a constant shift vector.

Every algebraic and measurable automorphism of the torus is continuous. Each continuous toral automorphism is a homeomorphism with zero periodic and constant parts and hence can be identified with an element of $SL(d, \mathbb{Z})$. And vice versa, each element of $SL(d, \mathbb{Z})$ uniquely determines a measurable, algebraic toral automorphism. Thus from now on the term *toral automorphism* will simply be reserved for elements of $SL(d, \mathbb{Z})$. We recall here that all Anosov diffeomorphisms on $\mathbb{T}^d$ are topologically conjugate to the toral automorphisms ([56], [86]).

Below we summarize a few definitions from ergodic theory along with some well known ergodic properties of toral automorphisms (cf. [66] p. 160, [67] and [10]).

Definition 3.1 *A dynamical system $(\mathbb{T}^d, \mu, F)$ is called a K-system (possesses K-property) if there exists subalgebra $\mathcal{A}$ of the algebra $\mathcal{M}$ of all μ-measurable sets such that*

- *i)* $\underset{n\in\mathbb{Z}^+}{\forall} \quad \mathcal{A} \subset F^n\mathcal{A}$
- *ii)* $\bigvee_{n\geq 0} F^n\mathcal{A} = \mathcal{M}$
- *ii)* $\bigwedge_{n\geq 0} F^{-n}\mathcal{A} = \mathcal{C}$

where $\mathcal{C}$ denotes the algebra of sets of measure O or 1. We note that the definition easily extends to abstract noncomutative version (in this case $\mathcal{C}$ stands for $c\mathbb{1}$).

Proposition 3.2 *Let F be a toral automorphism.*

The following statements are equivalent

a) no root of unity is an eigenvalue of F.

b) F is ergodic.

c) F is mixing.

d) F is a K-system.

In the sequel we will use the following result (cf. [126]).

Proposition 3.3 *The entropy $h(F)$ of any toral endomorphism F is computed by the formula*

$$h(F) = \sum_{|\lambda_j|\geq 1} \ln|\lambda_j|, \tag{3.1}$$

where λ_j denote the eigenvalues of A.

From the formula (3.1) one immediately sees that a toral automorphism has zero entropy iff all its eigenvalues are of modulus 1. In fact much stronger result holds (for proof see Section 3.5).

Proposition 3.4 *A toral automorphism has zero entropy iff all its eigenvalues are roots of unity. In particular all ergodic toral automorphisms have positive entropy.*

Given any toral automorphism F we denote by P its characteristic polynomial and by $\{P_1, ..., P_s\}$ the complete set of its distinct irreducible (over $\mathbb{Q}$) factors. Let d_j denote the degree of polynomial P_j and h_j the KS-entropy of any toral automorphism with the characteristic polynomial P_j. For each P_j we define its dimensionally averaged KS-entropy as

$$\hat{h}_j = \frac{h_j}{d_j}. \tag{3.2}$$

Definition 3.5 *Assuming the above notation, we define minimal dimensionally averaged entropy of F (denoted $\hat{h}(F)$) as*

$$\hat{h}(F) = \min_{j=1,\dots,s} \hat{h}_j$$

3.1.2 Main theorems

In this section we state two main theorems of Part I of this work. Both theorems concern the asymptotics of the dissipation time for toral automorphisms in arbitrary dimension. Our main task is to derive not only the logarithmic order of the asymptotics in the case of chaotic maps but also to find the exact value of the constant and relate it to dynamical properties of the map via the connection with the minimal dimensionally averaged KS-entropy of its irreducible blocks.

In order to be able to perform exact calculations we need to specify more concretely the family of noise kernels we are going to work with. Namely, we assume here that the noise kernel is α-stable, cf. (2.15), that is

$$g_{\epsilon,\alpha}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} e^{-|\epsilon\boldsymbol{k}|)^\alpha} \boldsymbol{e}_{\boldsymbol{k}}(\boldsymbol{x}), \tag{3.3}$$

Under this assumption we have the following results regarding the dissipation time of $\|T^n_{\epsilon,\alpha}\|$ (in order to emphasize that our calculations depend in an essential way on the

choice of the noise kernel we denote noisy operators associated with α-stable kennels by of $T_{\epsilon,\alpha}$)

Theorem 3.6 *Let F be any toral automorphism, U_F the Koopman operator associated with F, $G_{\epsilon,\alpha}$ α-stable noise operator and $T_{\epsilon,\alpha} = G_{\epsilon,\alpha} U_F$. Then*
i) $T_{\epsilon,\alpha}$ has simple dissipation time iff F is not ergodic.
ii) $T_{\epsilon,\alpha}$ has logarithmic dissipation time iff F is ergodic.
iii) If $T_{\epsilon,\alpha}$ has logarithmic dissipation time then the dissipation rate constant satisfies the following constraint

$$\frac{1}{\hat{h}(F)} \leq R_c \leq \frac{1}{\tilde{h}(F)},$$

where $\tilde{h}(F)$ is a positive constant satisfying $\tilde{h}(F) \leq \hat{h}(F)$.

The natural question arises, whether the lower bound for the dissipation rate constant given in the above theorem is best possible. The next theorem and its corollary provides a strong argument in favor of this conjecture.

Theorem 3.7 *If F is ergodic and diagonalizable then*

$$\tau_c \approx \frac{1}{\hat{h}(F)} \ln(1/\epsilon).$$

That is, the dissipation rate constant of $T_{\epsilon,\alpha}$ is given by

$$R_c = \frac{1}{\hat{h}(F)}.$$

In this case ergodicity of toral automorphisms is equivalent to hyperbolicity (in higher dimension it is not true, since there exists so called quasihyperbolic automorphisms, which although ergodic, possesses eigenvalues of modulus one [12, 108]). Two dimensional hyperbolic toral automorphisms are usually referred to as the *cat maps*.
Using Corollary 3.17 and applying Theorem 3.7 in two and three dimensions one gets the following

Corollary 3.8 (Cat maps) *Let F be any ergodic, two or three dimensional toral automorphism. Then*

$$\tau_c \approx \frac{1}{\hat{h}(F)} \ln(1/\epsilon),$$

We end this section with a remark that toral automorphisms provide a good example on which robustness of the dissipation time (i.e. its independence of unimportant details of the underlying conservative dynamics) can be tested. To this end we compare the dissipation time with another characteristics of chaoticity - the decay of correlations. As we will show in Section 3.3 the dissipation time has the same logarithmic asymptotics among general class of Anosov diffeomorphisms while the decorrelation may be exponential (generic Anosov case) or super-exponential depending on particular map.

Indeed, we illustrate this fact below by the following result (obtained in the similar way as the above theorems) on the decay of correlations for d-dimensional toral automorphisms (for a proof see Section 3.5).

Proposition 3.9 *Let F be a diagonalizable ergodic toral automorphism, U_F its Koopman operator and λ any constant such that $0 < \lambda < \hat{h}(F)$. Then for any $f, h \in L_0^2(\mathbb{T}^{2d})$ the correlation function for noisy dynamics generated by F and and any α-stable noise decays superexponentially i.e.*

$$C_{f,h}^{\epsilon}(n) = \langle \bar{f}, T_{\epsilon,\alpha}^{n} h \rangle \leq \|f\| \, \|h\| \, e^{-\epsilon^{\alpha} \lambda^{\alpha n}}$$

Moreover, let f, $h \in G_\epsilon(L_0^2(\mathbb{T}^{2d}))$ be smooth observables, where G_ϵ denotes Gaussian noise operator. Then the decay of correlations of unperturbed operator U_F i.e. of toral automorphism itself is still superexponential

$$C_{f,h}(n) = \langle \bar{f}, U_F h \rangle \leq \|G_\epsilon^{-1} f\| \, \|G_\epsilon^{-1} h\| \, e^{-\epsilon^{2} \lambda^{2n}}.$$

3.1.3 Asymptotic arithmetic minimization problem

In this section we prepare the ground for the proof of theorems stated in previous section. To this end we need to derive a concrete version of the general formula (2.56) obtained in Section 2.7 for any toral automorphism and arbitrary noise kernel. Using the fact that here we consider only α-stable kernels the formula (2.56) can be rewritten as follows

$$\|T^n_{\epsilon,\alpha}\| \quad = \quad \max_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \prod_{l=1}^{n} \hat{g}_\epsilon(A^l\boldsymbol{k}) = e^{-\epsilon^\alpha \min_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \sum_{l=1}^n |A^l\boldsymbol{k}|^\alpha}. \tag{3.4}$$

Hence, for toral automorphisms and α-stable kernels, the calculation of the dissipation time reduces to the following nonlinear, asymptotic (large n) arithmetic minimization problem

$$\min_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \sum_{l=1}^{n} |A^l\boldsymbol{k}|^\alpha, \tag{3.5}$$

where $A \in SL(d,\mathbb{Z})$. When A is not ergodic the asymptotics of (3.5) is clearly of the order $O(n)$. Thus we will be only concerned with the ergodic case. For $d = 2$ the problem (3.5) can be solved easily as follows. Consider first the case that A is symmetric and $\alpha = 2$. From $\det(A) = 1$ we see that eigenvalues are λ, λ^{-1} with $|\lambda| > 1$. We have

$$\begin{aligned}
&\min_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \sum_{l=1}^{2n+1} |A^l\boldsymbol{k}|^2 = \min_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \sum_{l=-n}^{n} |A^l\boldsymbol{k}|^2 \\
= \quad &\min_{0\neq \boldsymbol{k}=\boldsymbol{k}_1+\boldsymbol{k}_2\in\mathbb{Z}^d} \left(|\boldsymbol{k}|^2 + \sum_{l=1}^{n} |\lambda|^{2l}|\boldsymbol{k}_1|^2 + |\lambda|^{-2l}|\boldsymbol{k}_2|^2 + \sum_{l=1}^{n} |\lambda|^{-2l}|\boldsymbol{k}_1|^2 + |\lambda|^{2l}|\boldsymbol{k}_2|^2 \right) \\
= \quad &\min_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \sum_{l=-n}^{n} |\lambda|^{2l}|\boldsymbol{k}|^2 = \sum_{l=-n}^{n} |\lambda|^{2l}.
\end{aligned}$$

Hence there exist constants C_1 and C_2 such that

$$C_1 e^{h(A)n} \leq \min_{0\neq \boldsymbol{k}\in\mathbb{Z}^d} \sum_{l=1}^{n} |A^l\boldsymbol{k}|^2 \leq C_2 e^{h(A)n}.$$

where $h(A)$ denotes the KS-entropy of A. The estimates for the general case of non-symmetric A and $\alpha \neq 2$ are similar.

In higher dimensions, the solution to (3.5) is much more involved because of the presence of different eigenvalues with absolute values bigger than one. We have the following general estimate

Theorem 3.10 *Let $A \in SL(d, \mathbb{Z})$ be ergodic. There exist constants C_1 and C_2 such that for any $0 < \delta < 1$ and sufficiently large n*

$$C_1 e^{(1-\delta)\alpha \tilde{h}(A) n} \leq \min_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \boldsymbol{k}|^\alpha \leq C_2 n e^{\alpha \hat{h}(A) n} \tag{3.6}$$

where as before $\hat{h}(A)$ denotes minimal dimensionally averaged entropy of A and $\tilde{h}(A)$ denotes a constant satisfying $0 < \tilde{h}(A) \leq \hat{h}(A)$, with equality achieved for all diagonalizable matrices A.

The question whether the equality $\tilde{h}(A) = \hat{h}(A)$ holds for all ergodic matrices remains open.

The proof of the theorem relies on nontrivial use of three number-theoretical results stated below.

I. Minkowski's Theorem on linear forms

Let $L_1, ..., L_d$ be linearly independent linear forms on $\mathbb{R}^d$ which are real or occur in conjugate complex pairs. Suppose $a_1, a_2, ..., a_d$ are real positive numbers satisfying $a_1 a_2 ... a_d = 1$ and $a_i = a_j$, whenever $L_i = \bar{L}_j$. Then there exists a nonzero integer vector $\boldsymbol{k} \in \mathbb{Z}^d$ such that for every $j = 1, ..., d$,

$$|L_j \boldsymbol{k}| \leq D a_j, \tag{3.7}$$

where $D = |\det[L_1, ..., L_d]|^{1/d}$.

Minkowski's Theorem on linear forms will be used to obtain a sharp upper bound on the asymptotic solution of the arithmetic minimization problem. The proof of the

above theorem and its generalization to arbitrary lattices can be found in [99] (Chap. VI).

II. Schmidt's Subspace Theorem

Let $L_1, ..., L_d$ be linearly independent linear forms on $\mathbb{R}^d$ with real or complex algebraic coefficients. Given $\delta > 0$, there are finitely many proper rational subspaces of $\mathbb{R}^d$ such that every nonzero integer vector $\boldsymbol{k}$ with

$$\prod_{j=1}^{d} |L_j \boldsymbol{k}| < |\boldsymbol{k}|^{-\delta} \tag{3.8}$$

lies in one of these subspaces.

Schmidt's Subspace Theorem will be used in conjunction with Van der Waerden's Theorem on arithmetic progressions (see below) to obtain a sharp lower bound for the asymptotic solution of the arithmetic minimization problem. The proof of Schmidt's Subspace Theorem can be found in [117] (Theorem 1F, p. 153).

Definition 3.11 *For a given set of linear forms and for fixed $\delta > 0$, the smallest collection of proper rational subspaces of $\mathbb{R}^d$ which contain all nonzero integer vectors satisfying (3.8), is called the exceptional set and denoted by E_δ.*

A main difficulty to be resolved in using Schmidt's Subspace Theorem is to show that the minimizer of either the original problem (3.5) or an equivalent problem does not lie in the respective exceptional set which is in general unknown. We will pursue the latter route by using Van der Waerden's Theorem on arithmetic progressions to show that one can always construct an equivalent minimization problem whose minimizer is guaranteed to lie outside the corresponding exceptional set. To this end we note that Schmidt's Subspace Theorem is true when the standard lattice $\mathbb{Z}^d$ is replaced by any other rational lattice, that is any lattice of the form $\Lambda = Q(\mathbb{Z}^d)$ where $Q \in GL(d, \mathbb{Q})$. Schmidt's subspace theorem can be generalized to this situation by considering the set of new forms $\tilde{L}_j = L_j Q$. The fact that $Q \in GL(d, \mathbb{Q})$ implies immediately that $\tilde{L}_j$ are still linearly independent forms on $\mathbb{R}^d$ with real or complex algebraic coefficients.

III. Van der Waerden's Theorem on arithmetic progressions

Let k and d be two arbitrary natural numbers. Then there exists a natural number $n_*(k,d)$ such that, if an arbitrary segment of length $n \geq n_*$ of the sequence of natural numbers is divided in any manner into k (finite) subsequences, then an arithmetic progression of length d appears in at least one of these subsequences.

The original proof was published in [124]; Lukomskaya's simplification can be found in [83].

Before presenting the proof of our main results we state a number of technical facts concerning the structure of toral automorphisms.

3.1.4 Algebraic structure of toral automorphisms

In this section we denote by $GL(d,\mathbb{Q})$ the group of nonsingular $d \times d$ matrices with rational entries or the group of linear operators on Euclidean space $\mathbb{R}^d$, which are represented in standard basis by such matrices. We generally use the same symbol to denote both operator and its matrix.

In the sequel a vector $x \in \mathbb{R}^d$ will be called an integer (or integral) vector if all its components are integers, and similarly a rational, an algebraic vector if all its components are rational or respectively algebraic numbers. The term *rational subspace of* $\mathbb{R}^d$ will then refer to a linear subspace of $\mathbb{R}^d$ spanned by rational vectors (cf. [117] p. 113).

Definition 3.12 *$A \in GL(d,\mathbb{Q})$ is called irreducible (over $\mathbb{Q}$) if its characteristic polynomial is irreducible in $\mathbb{Q}[x]$.*

Lemma 3.13 *The following statements about a matrix $A \in GL(d,\mathbb{Q})$ are equivalent.*

a) A is irreducible.

b) A does not possess any proper rational A-invariant subspaces of $\mathbb{R}^d$.

c) No rational proper subspace of $\mathbb{R}^d$ *is contained in any proper* A*-invariant subspace of* $\mathbb{R}^d$.

d) For any nonzero $\boldsymbol{q} \in \mathbb{Q}^d$ *and any arithmetic progression of integer numbers* $n_1, ..., n_d$*, the set* $\{A^{n_1}\boldsymbol{q}, A^{n_2}\boldsymbol{q}, ..., A^{n_d}\boldsymbol{q}\}$ *forms a basis of* $\mathbb{R}^d$.

e) $A^\dagger$ *is irreducible.*

f) No nonzero $\boldsymbol{q} \in \mathbb{Q}^d$ *is orthogonal to any proper* A*-invariant subspace of* $\mathbb{R}^d$.

g) No proper A*-invariant subspace of* $\mathbb{R}^d$ *is contained in any proper rational subspace of* $\mathbb{R}^d$.

Definition 3.14 *We say that operator* $A \in GL(d, \mathbb{Q})$ *is completely decomposable over* $\mathbb{Q}$ *if there exists a rational basis of* $\mathbb{R}^d$ *in which* A *admits the following block diagonal form*

$$\begin{bmatrix} A_1 & 0 & ... & 0 \\ 0 & A_2 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & A_r \end{bmatrix}, \tag{3.9}$$

where for each $j = 1, ..., r \leq d$*,* $A_j \in GL(d_j, \mathbb{Q})$ *is irreducible and* $\sum_{j=1}^{r} d_j = d$.

In general, any matrix $A \in GL(d, \mathbb{Q})$ admits a rational block diagonal representation $[A_j]_{j=1,...,r}$. The smallest rational blocks to which A can be decomposed are called elementary divisor blocks. The characteristic polynomial corresponding to any elementary divisor block is of the form p^m, where p is an irreducible (over $\mathbb{Q}$) polynomial (see, e.g., [46]). Although elementary divisor blocks cannot be decomposed over $\mathbb{Q}$ into smaller invariant blocks, some elementary divisor blocks may not be irreducible. This happens iff $m > 1$ iff A is not completely decomposable over $\mathbb{Q}$. One has the following elementary fact (see Section 3.5 for a proof).

Proposition 3.15 $A \in GL(d, \mathbb{Q})$ *is completely decomposable over* $\mathbb{Q}$ *iff* A *is diagonalizable.*

However, even if $A \in GL(d, \mathbb{Q})$ is not completely decomposable, each elementary divisor block of A can be uniquely represented (in a rational basis) in the following block upper triangular form

$$\begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{3.10}$$

where B is the unique rational irreducible sub-block associated with A-invariant rational subspace of that elementary divisor and C, D denote some rational matrices.

Proposition 3.16 *All the eigenvalues of an irreducible matrix $A \in GL(d, \mathbb{Q})$ are distinct (complex) algebraic numbers. In particular all irreducible matrices are diagonalizable.*

The proofs of the above propositions can be found in Appendix B.

Finally we note that since the leading coefficient and constant term of a characteristic polynomial of any toral automorphism are equal to 1, the only possible rational eigenvalues of such map are ± 1 or $\pm i$. The latter fact implies that ergodic toral automorphisms do not possesses rational eigenvalues. Thus we have the following

Corollary 3.17 *Let F be an ergodic, two or three dimensional toral automorphism. Then F is irreducible (and hence diagonalizable).*

3.1.5 The Proof of Theorems 3.6, 3.7 and 3.10

This section is entirely devoted to the proofs of the main theorems of this chapter. We start with the proof of Theorem 3.10, which constitutes the main ingredient in the proofs of Theorems 3.6 and 3.7.

Let $[A_j]_{j=1,\ldots,r}$ be a rational block-diagonal decomposition of A into elementary divisor blocks. Since $A \in SL(d, \mathbb{Z})$, there exist a transition matrix $Q \in SL(d, \mathbb{Q})$ such that for every $l \in \mathbb{Z}$,

$$A^l = Q^{-1}([A_j])^l Q$$

and moreover each elementary divisor block $[A]_j$ is represented in its block upper triangular form (3.10).

The matrix Q defines a new lattice $\Lambda = Q(\mathbb{Z}^d)$ and acts bijectively between this lattice and the standard lattice $\mathbb{Z}^d$. Hence

$$\min_{0\neq \boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|A^l\boldsymbol{k}|^\alpha = \min_{0\neq \boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|Q^{-1}([A_j])^lQ\boldsymbol{k}|^\alpha = \min_{0\neq \boldsymbol{q}\in\Lambda}\sum_{l=1}^{n}|Q^{-1}([A_j])^l\boldsymbol{q}|^\alpha.$$

Moreover

$$\|Q\|^{-\alpha}|([A_j])^l\boldsymbol{q}|^\alpha \leq |Q^{-1}([A_j])^l\boldsymbol{q}|^\alpha \leq \|Q^{-1}\|^\alpha|([A_j])^l\boldsymbol{q}|^\alpha, \quad \forall l, j, \alpha.$$

Now we decompose Λ into the direct sum of lower dimensional sublattices Λ_j corresponding to invariant blocks $[A_j]$. So that

$$\min_{0\neq \boldsymbol{q}\in\Lambda}\sum_{l=1}^{n}|([A_j])^l\boldsymbol{q}|^\alpha = \min_{j\in\{1,\ldots,r\}}\min_{0\neq \boldsymbol{q}\in\Lambda_j}\sum_{l=1}^{n}|(A_j)^l\boldsymbol{q}|^\alpha. \tag{3.11}$$

Thus, without loss of generality, we may specialize to the case that A is already indecomposable over $\mathbb{Q}$ i.e. A does not possesses any proper elementary divisor blocks. To simplify the notation we will work with the standard lattice $\Lambda = \mathbb{Z}^d$. According to the remarks following the statements of Minkowski's and Schmidt's Theorems the proof can be easily adapted for any rational lattice $\Lambda = Q(\mathbb{Z}^d)$.

Since the technique of the proof differs depending on diagonalizability of A we consider two cases:

Diagonalizable case

Here we concentrate on the case when A is diagonalizable and hence due to its indecomposability irreducible (cf. Proposition 3.15).

We denote by λ_j ($j = 1, ..., d$) the eigenvalues of A. Following Proposition 3.16 we note that λ_j are distinct (possibly complex) algebraic numbers and hence there exists a basis (of $\mathbb{C}^d$) $\{\boldsymbol{v}_j\}_{j=1,\ldots,d}$ composed of normalized algebraic eigenvectors corresponding to eigenvalues λ_j.

We denote by $[P_j]_{j=1}^d$ the projections on $[\boldsymbol{v}_j]$, and by $[L_j]$ the corresponding linear forms. It is easy to check that $[L_j]$ are given, in the Riesz identification, by the eigenvectors $[\boldsymbol{u}_j]$ of the matrix $A^\dagger$ which are co-orthogonal to $[\boldsymbol{v}_j]$, i.e., $\langle \boldsymbol{u}_i, \boldsymbol{v}_j \rangle = 0$ for $i \neq j$. $[\boldsymbol{u}_j]$ and $[\boldsymbol{v}_j]$ are real or occur in complex conjugate pairs. We have

$$\boldsymbol{x} = \sum_{j=1}^d P_j \boldsymbol{x} = \sum_{j=1}^d (L_j \boldsymbol{x}) \boldsymbol{v}_j = \sum_{j=1}^d \langle \boldsymbol{x}, \boldsymbol{u}_j \rangle \boldsymbol{v}_j, \quad \forall \boldsymbol{x} \in \mathbb{R}^d.$$

The equivalence between any two norms in a finite dimensional vector space, implies the existence of absolute constants C_1, C_2 such that

$$C_1 \sum_{j=1}^d |P_j \boldsymbol{x}|^2 \leq |\boldsymbol{x}|^2 \leq C_2 \sum_{j=1}^d |P_j \boldsymbol{x}|^2.$$

Using the above inequalities, the monotonicity of a map $\boldsymbol{x} \mapsto \boldsymbol{x}^\alpha$ and an obvious inequality $(a+b)^\alpha \leq a^\alpha + b^\alpha$, which holds for all positive a, b and $\alpha \in (0,1]$ one obtains the following estimates

$$\begin{aligned} \sum_{l=1}^n |A^l \boldsymbol{k}|^\alpha &\leq \sum_{l=1}^n \left(C_2 \sum_{j=1}^d |P_j A^l \boldsymbol{k}|^2 \right)^\alpha = C_2^\alpha \sum_{l=1}^n \left(\sum_{j=1}^d |\lambda_j|^{2l} |P_j \boldsymbol{k}|^2 \right)^\alpha \\ &\leq C_2^\alpha \sum_{l=1}^n \sum_{j=1}^d |\lambda_j|^{\alpha l} |P_j \boldsymbol{k}|^\alpha = C_2^\alpha \sum_{j=1}^d \left(\sum_{l=1}^n |\lambda_j|^{\alpha l} \right) |P_j \boldsymbol{k}|^\alpha \end{aligned}$$

and on the other hand

$$\begin{aligned} \sum_{l=1}^n |A^l \boldsymbol{k}|^\alpha &\geq \left(\sum_{l=1}^n |A^l \boldsymbol{k}|^2 \right)^\alpha \geq \left(\sum_{l=1}^n C_1 \sum_{j=1}^d |P_j A^l \boldsymbol{k}|^2 \right)^\alpha \\ &= C_1^\alpha \left(\sum_{l=1}^n \sum_{j=1}^d |\lambda_j|^{2l} |P_j \boldsymbol{k}|^2 \right)^\alpha = C_1^\alpha \left(\sum_{j=1}^d \left(\sum_{l=1}^n |\lambda_j|^{2l} \right) |P_j \boldsymbol{k}|^2 \right)^\alpha. \end{aligned}$$

Now we introduce some notation

$$\hat{\lambda}_j := \max\{1, |\lambda_j|\}, \tag{3.12}$$

$$\hat{\lambda}_{geo} := \left(\prod_{j=1}^d \hat{\lambda}_j \right)^{1/d}. \tag{3.13}$$

One can easily observe that there exists a constant C such that

$$C\hat{\lambda}_j^{\alpha n} \leq \sum_{l=1}^{n} |\lambda_j|^{\alpha l} \leq n\hat{\lambda}_j^{\alpha n}.$$

In the sequel we do not distinguish between particular values of constants appearing in computations. The symbols $C_1, C_2, ..$ are used to denote any generic constants independent of n.

The normalization condition $|\boldsymbol{v}_j| = 1$ implies the following relation

$$|P_j\boldsymbol{x}| = |L_j\boldsymbol{x}|. \tag{3.14}$$

Combining the above estimates one gets the following general bounds

$$C_1 \left(\sum_{j=1}^{d} \hat{\lambda}_j^{2n} |L_j\boldsymbol{k}|^2 \right)^{\alpha} \leq \sum_{l=1}^{n} |A^l\boldsymbol{k}|^{\alpha} \leq C_2 n \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j\boldsymbol{k}|^{\alpha}. \tag{3.15}$$

Therefore in order to estimate (3.5) it suffices, essentially, to estimate

$$\min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d} \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j\boldsymbol{k}|^{\alpha}. \tag{3.16}$$

We denote by $\boldsymbol{z}_n$ the sequence of minimizers i.e. nonzero integral vectors solving (3.16).

Upper bound.

For the upper bound we assign to the set of linear forms L_j the set $\mathcal{A}$ composed of all real vectors $\boldsymbol{a} = (a_1, ..., a_d)$ satisfying the conditions $a_j > 0$, for $j = 1, ..., d$ and $a_i = a_j$ whenever $L_i = \bar{L}_j$ and

$$\prod_{j=1}^{d} a_j = 1. \tag{3.17}$$

From Minkowski's theorem on linear forms, we know that for any $\boldsymbol{a} \in \mathcal{A}$, there exists nonzero integral vector $\boldsymbol{k}_{\boldsymbol{a}}$ satisfying $|L_j\boldsymbol{k}_{\boldsymbol{a}}| \leq Da_j$, $j = 1, ..., d$, where $D = |\det[L_1, ..., L_d]|^{1/d}$.

Thus

$$\sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j \boldsymbol{k}_{\boldsymbol{a}}|^{\alpha} \leq D \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} a_j^{\alpha}. \tag{3.18}$$

The minimizing property of $\boldsymbol{z}_n$ implies that for any $\boldsymbol{a} \in \mathcal{A}$,

$$\sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j \boldsymbol{z}_n|^{\alpha} \leq \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j \boldsymbol{k}_{\boldsymbol{a}}|^{\alpha}. \tag{3.19}$$

Thus combining (3.18) and (3.19), and applying the Lagrange multipliers minimization with the constraint (3.17) (and using the fact that $\hat{\lambda}_i = \hat{\lambda}_j$ whenever $L_i = \bar{L}_j$), we get

$$\sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j \boldsymbol{z}_n|^{\alpha} \leq D \min_{\boldsymbol{a} \in \mathcal{A}} \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} a_j^{\alpha} = dD \left(\prod_{j=1}^{d} \hat{\lambda}_j^{\alpha n} \right)^{1/d} = dD \hat{\lambda}_{geo}^{2\alpha n}. \tag{3.20}$$

Thus the following upper bound holds

$$\min_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \boldsymbol{k}|^{\alpha} \leq C_2 n \hat{\lambda}_{geo}^{2\alpha n}. \tag{3.21}$$

Lower bound.

Let m denote an arbitrary natural number. Using the fact that A acts bijectively on $\mathbb{Z}^d$ we can restate the minimization problem (3.16) in the following form

$$\begin{aligned} \min_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j \boldsymbol{k}|^{\alpha} &= \min_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j A^{-m} A^m \boldsymbol{k}|^{\alpha} && (3.22) \\ &= \min_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |\lambda_j|^{-\alpha m} |L_j A^m \boldsymbol{k}|^{\alpha} && (3.23) \end{aligned}$$

That is

$$\sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j \boldsymbol{z}_n|^{\alpha} = \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |\lambda_j|^{-\alpha m} |L_j A^m \boldsymbol{z}_n|^{\alpha}. \tag{3.24}$$

We choose arbitrary $\delta > 0$ and consider the exceptional set E_δ (see Definition 3.11) associated with the system of linear forms $[L_j]$. Since $[L_j]$ correspond to the eigenpairs $[\bar{\lambda}_j, \boldsymbol{u}_j]$ of $A^\dagger$ they are linearly independent linear forms with (real or complex)

algebraic coefficients. Thus the subspace theorem asserts that E_δ is a finite collection of proper rational subspaces of $\mathbb{R}^d$. We denote by k_δ the number of subspaces forming E_δ.

Now we want to show that for all sufficiently large n there exist an integer $m \leq n$ such that $A^m \boldsymbol{z}_n$ does not lie in any element of E_δ. To this end we assume to the contrary that all $A^m \boldsymbol{z}_n$ lie in the subspaces forming E_δ and we divide the sequence of natural numbers $1, ..., n$ into k_δ classes in such a way that two numbers m_1 and m_2 are in the same class if $A^{m_1}\boldsymbol{z}_n$ and $A^{m_2}\boldsymbol{z}_n$ lie in the same element of E_δ. Now let $n_*(k_\delta, d)$ be the number given in the van der Waerden theorem and let $n \geq n_*$. Then there exists an arithmetic progression $m_1, ..., m_d$ in one of these subsequences. By Lemma 3.13 d) the set of vectors $\{A^{m_1}\boldsymbol{z}_n, A^{m_2}\boldsymbol{z}_n, ..., A^{m_d}\boldsymbol{z}_n\}$ forms a basis of the whole space $\mathbb{R}^d$, which contradicts the fact that they lie in one fixed rational proper subspace. Hence for any $\delta > 0$ and $n \geq n_*$ there exists $m_* \leq n$ such that $A^{m_*}\boldsymbol{z}_n$ does not lie in any element of E_δ.

Now, introducing the notation

$$\hat{\boldsymbol{z}}_n = A^{m_*}\boldsymbol{z}_n \tag{3.25}$$

one concludes from (3.24) that for any $\delta > 0$ and all $n \geq n_*$ the following equality and estimate hold

$$\sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |L_j \boldsymbol{z}_n|^\alpha = \sum_{j=1}^{d} \hat{\lambda}_j^{\alpha n} |\lambda_j|^{-\alpha m_*} |L_j \hat{\boldsymbol{z}}_n|^\alpha \tag{3.26}$$

$$\prod_{j=1}^{d} |L_j \hat{\boldsymbol{z}}_n| \geq \frac{1}{|\hat{\boldsymbol{z}}_n|^\delta}. \tag{3.27}$$

Inequality (3.27) may be rewritten as

$$\prod_{j=1}^{d} |L_j \hat{\boldsymbol{z}}_n| = \frac{1}{f(|\hat{\boldsymbol{z}}_n|)^\delta} \tag{3.28}$$

with some $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(r) \leq r, \forall r > 0$.

Using (3.25) and (3.20) we obtain the existence of a constant $\lambda > 1$ such that

$$\begin{aligned} f(|\hat{\boldsymbol{z}}_n|) &\leq |\hat{\boldsymbol{z}}_n| = |A^{m_*}\boldsymbol{z}_n| \leq \hat{\lambda}_{max}^{m_*}|\boldsymbol{z}_n| \\ &\leq \hat{\lambda}_{max}^{n}\sum_{j=1}^{d}\hat{\lambda}_j^n|L_j\boldsymbol{z}_n| \leq dD(\hat{\lambda}_{max}\hat{\lambda}_{geo})^n \leq \lambda^n. \end{aligned} \tag{3.29}$$

Note that $\prod_j \lambda_j = 1$. So, by (3.28) the quantities $B_{j,n} = \left(|\lambda_j|^{-m_*} f(|\hat{\boldsymbol{z}}_n|)^{\delta/d}|L_j\hat{\boldsymbol{z}}_n|\right)^{\alpha}$, $j = 1, ..., d$ satisfy the constraint

$$\prod_{j=1}^{d} B_{j,n} = 1, \quad \forall n > n_*. \tag{3.30}$$

Thus applying (3.29) and the Lagrange multipliers minimization with the constraint (3.30) one gets

$$\sum_{j=1}^{d}\hat{\lambda}_j^{\alpha n}|\lambda_j|^{-\alpha m_*}|L_j\hat{\boldsymbol{z}}_n|^{\alpha} = f(|\hat{\boldsymbol{z}}_n|)^{-\alpha\delta/d}\sum_{j=1}^{d}\hat{\lambda}_j^{\alpha n}B_{j,n} \geq \lambda^{-2\alpha n\delta/d}\hat{\lambda}_{geo}^{\alpha n} =: \hat{\lambda}_{geo}^{\alpha n(1-\hat{\delta})}.$$

This and equality (3.26) yields the following lower bound for (3.16)

$$\min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{j=1}^{d}\hat{\lambda}_j^{\alpha n}|L_j\boldsymbol{k}|^{\alpha} \geq \hat{\lambda}_{geo}^{2\alpha n(1-\hat{\delta})}. \tag{3.31}$$

Non-diagonalizable case

We move on to the general case where A is not irreducible (but, as assumed at the beginning of the proof, indecomposable over $\mathbb{Q}$). We denote by B the invariant irreducible sub-block of A given by its block upper triangular decomposition (3.10) and by S the rational invariant subspace associated with this block. We note that B as an irreducible matrix is diagonalizable.

Upper bound.

Note that

$$\min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|A^l\boldsymbol{k}|^{\alpha} \leq \min_{0\neq\boldsymbol{k}\in S\cap\mathbb{Z}^d}\sum_{l=1}^{n}|B^l\boldsymbol{k}|^{\alpha}. \tag{3.32}$$

The corresponding upper bound (3.21) for B is then also an upper bound for the whole matrix A. We note that geometric average of $\hat{\lambda}_j$ over S is equal to the geometric average of all $\hat{\lambda}_j$ associated with matrix A (i.e. over the whole space $\mathbb{R}^d$).

Lower bound.

According to our assumption A is indecomposable and thus the characteristic polynomial of A is of the form p^m for some irreducible p. All Jordan blocks of A have the same size m and different Jordan blocks correspond to distinct eigenvalues. Denote by b the number of the Jordan blocks in A and by λ_j, where $j = 1, .., l$ all these distinct eigenvalues . Since each λ_j has algebraic multiplicity m, we get $d = mb$. Let $\{\boldsymbol{v}_{j,h}\}_{j=1,\dots,b;h=0,\dots,m-1}$ be a basis (of $\mathbb{C}^d$) in which A admits the Jordan canonical form. As usually $L_{j,h}$ will denote the corresponding linear forms. Each $\boldsymbol{v}_{j,h}$ can be regarded as a generalized eigenvector of A associated with an eigenvalue λ_j. We assume that these generalized eigenvectors are ordered according to their degree i.e. $\boldsymbol{v}_{j,h}$ satisfies the equation $(A - \lambda_j I)^{1+h}\boldsymbol{v}_{j,h} = 0$. Reordering the eigenvalues, if necessary, we can also assume that λ_1 has the largest modulus among all eigenvalues of A and hence $\hat{\lambda}_1 = |\lambda_1|$. Let $\boldsymbol{z}_n$ be the sequence of minimizers solving (3.5). We first note that for each n there exists $0 \leq h \leq m - 1$ such that $L_{1,h}\boldsymbol{z}_n \neq 0$. Indeed, otherwise for all $h = 0, ..., m - 1$, $L_{1,h}\boldsymbol{z}_n = 0$ and consequently for any n and h $L_{1,h}A^n\boldsymbol{z}_n = 0$. The latter implies that the set of consecutive iterations $\{\boldsymbol{z}_n, A^1\boldsymbol{z}_n, A^2\boldsymbol{z}_n, ...\}$ spans a proper rational A-invariant subspace of $\mathbb{R}^d$ which does not have any intersection with the subspace spanned by the generalized eigenvectors of A associated with eigenvalue λ_1. This clearly contradicts the irreducibility of p. Now, for given n we denote by $h(n)$ the biggest index h for which the condition $L_{1,h}\boldsymbol{z}_n \neq 0$ holds.

We have the following estimate

$$\hat{\lambda}_1^{\alpha n}|L_{1,h(n)}\boldsymbol{z}_n|^\alpha \leq \left(\sum_{j=1}^{b}\sum_{h=0}^{m-1}\left|\sum_{i=0}^{m-1-h}\lambda_j^{n-i}\binom{n}{i}L_{j,h+i}\boldsymbol{z}_n\right|^2\right)^\alpha \tag{3.33}$$

$$\leq \quad C_1|A^n\boldsymbol{z}_n|^\alpha \leq C_1\sum_{l=1}^{n}|A^l\boldsymbol{z}_n|^\alpha \leq C_2 n\hat{\lambda}_{geo}^{\alpha n}, \tag{3.34}$$

where the last inequality follows from previously established upper bound.

From the Diophantine approximation and the assumption that $|L_{1,h(n)}\boldsymbol{z}_n| \neq 0$, there exists $\beta > 0$ such that (see [117] p. 164)

$$|L_{1,h(n)}\boldsymbol{z}_n| \geq \frac{1}{|\boldsymbol{z}_n|^\beta}. \tag{3.35}$$

Thus combining (3.33) with (3.35) one gets

$$\hat{\lambda}_1^{\alpha n}|\boldsymbol{z}_n|^{-\alpha\beta} \leq \hat{\lambda}_1^{\alpha n}|L_{1,h(n)}\boldsymbol{z}_n|^\alpha \leq C_2 n\hat{\lambda}_{geo}^{\alpha n}.$$

After rearrangements one obtains the following lower bound estimate for (3.5)

$$\frac{C}{n}\hat{\lambda}^{\alpha n} \leq C|\boldsymbol{z}_n|^\alpha \leq \sum_{l=1}^{n}|A^l\boldsymbol{z}_n|^\alpha = \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|A^l\boldsymbol{k}|^\alpha, \tag{3.36}$$

where

$$\hat{\lambda} = \left(\frac{\hat{\lambda}_1}{\hat{\lambda}_{geo}}\right)^{1/\beta}.$$

We note that ergodicity of A implies $\hat{\lambda}_1 > \hat{\lambda}_{geo} > 1$ (see (3.13), (3.1) and Proposition 3.4) which ensures non-triviality of this lower bound.

Now in order to finish the proof is suffices to combine the estimates (3.21), (3.31) and (3.36), and note that

$$\hat{\lambda}_{geo}^{\alpha n} = e^{\alpha\frac{h(A)}{d}n} = e^{\alpha\hat{h}(A)n}$$

which yields (3.6). ■

Proofs of Theorems 3.6 and 3.7

We start with part i) of Theorem 3.6 which follows as a simple consequence of Theorem 2.12. Indeed, it suffices to construct an eigenfunction of U_F which belongs to $L^2_0(\mathbb{T}^d)\cap H^\alpha(\mathbb{T}^d)$. Directly from Proposition 3.2 one concludes that F, and hence also A, possesses a root of unity in its spectrum. This means that $A^m\boldsymbol{k}_0 = \boldsymbol{k}_0$, for some m and certain nonzero vector $\boldsymbol{k}_0$, which can be chosen to be an integer. Now we define

$$f = \boldsymbol{f}_{\boldsymbol{k}_0} + \boldsymbol{f}_{A\boldsymbol{k}_0} + ... + \boldsymbol{f}_{A^{m-1}\boldsymbol{k}_0}$$

Obviously $f \in L^2_0(\mathbb{T}^d)\cap H^\alpha(\mathbb{T}^d)$, for any α. To complete the proof it suffices to notice that

$$U_F f = \boldsymbol{f}_{A\boldsymbol{k}_0} + \boldsymbol{f}_{A^2\boldsymbol{k}_0} + ... + \boldsymbol{f}_{A^m\boldsymbol{k}_0} = \boldsymbol{f}_{\boldsymbol{k}_0} + \boldsymbol{f}_{A\boldsymbol{k}_0} + ... + \boldsymbol{f}_{A^{m-1}\boldsymbol{k}_0} = f. \qquad \blacksquare$$

Now we apply Theorem 3.10 to prove part ii) and iii) of Theorem 3.6 and Theorem 3.7.

In order to determine the dissipation time of $T_{\epsilon,\alpha}$ one has to determine the asymptotics of $\|T^n_{\epsilon,\alpha}\|$ when n goes to infinity. According to formula (3.4) this problem reduces to problem (3.5) which has been solved by Theorem 3.10). This in view of this theorem there exist constants C_1 and C_2 such that for any $\delta, \delta' > 0$ and sufficiently large n

$$C_1 e^{(1-\delta)\alpha\tilde{h}(A)n} \le \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d} \sum_{l=1}^{n} |A^l\boldsymbol{k}|^\alpha \le C_2 n e^{\alpha\hat{h}(A)n} \le C_2 e^{(1+\delta')\alpha\hat{h}(A)n}$$

Using formula (3.4)

$$e^{-\epsilon^\alpha C_2 e^{(1+\delta')\alpha\hat{h}(A)n}} \le \|T^n_{\epsilon,\alpha}\| \le e^{-\epsilon^\alpha C_1 e^{(1-\delta)\alpha\tilde{h}(A)n}}.$$

Now when $n = \tau_c$, we have

$$C_1 e^{(1-\delta)\tilde{h}(A)\tau_c} \le \frac{1}{\epsilon} \le C_2 e^{(1+\delta')\hat{h}(A)\tau_c}$$

and

$$\frac{1}{(1+\delta')\hat{h}(A)}\Big(\ln(1/\epsilon) - \ln C_2\Big) \le \tau_c \le \frac{1}{(1-\delta)\tilde{h}(A)}\Big(\ln(1/\epsilon) - \ln C_1\Big),$$

which proves part ii) of Theorem 3.6 i.e. the logarithmic growth of dissipation time as a function of ϵ^{-1}.

Moreover, using the definition of dissipation rate constant

$$R_c = \lim_{\epsilon \to 0} \frac{\tau_c}{\ln(1/\epsilon)}$$

we obtain

$$\frac{1}{(1+\delta')\hat{h}(A)} \leq R_c \leq \frac{1}{(1-\delta)\tilde{h}(A)}.$$

Finally letting $\delta \to 0$ and $\delta' \to 0$ we arrive at the following results:

- The general case - Theorem 3.6 iii)

$$\frac{d}{\hat{h}(F)} \leq R_c \leq \frac{1}{\tilde{h}(F)}$$

- The diagonalizable case - Theorem 3.7

$$R_c = \frac{1}{\hat{h}(F)}.$$

This completes the proof. ■

3.2 Generalizations and Applications

In this section we generalize or apply our main results i.e. Theorems 3.6 and 3.7 in the following situations

1. Coarse grained dynamics (Section 3.2.1).
2. Affine toral maps (Section 3.2.2).
3. Degenerate noises (Section 3.2.3).
4. Kinematic dynamo problem (Section 3.2.4).

3.2.1 Dissipation time of coarse-grained dynamics

The uncertainties in the initial preparation and the final measurement of the noiseless system give rise to non-cumulative random perturbations to the system. The dynamics of such systems can be modeled by the coarse-grained family of noisy operators $\tilde{T}^{(n)}_{\epsilon,\alpha}$ introduced in Section 2.1.4. In this section we compute corresponding dissipation time $\tilde{\tau}_c$. We show that, remarkably, for ergodic toral automorphisms both dissipation times are asymptotically the same, despite considerable difference in the structure of the corresponding noisy dynamical systems.

To prove this result one can represent the action of U_F or more generally U_F^n in the Fourier series along the lines introduced in Section refOpt

$$U_F^n \boldsymbol{e}_{\boldsymbol{k}} = \sum_{0\neq \boldsymbol{k}'\in\mathbb{Z}^d} u^{(n)}_{\boldsymbol{k},\boldsymbol{k}'} \boldsymbol{e}_{\boldsymbol{k}'},$$

where $u^{(1)}_{\boldsymbol{k},\boldsymbol{k}'}$ coincides with $u_{\boldsymbol{k},\boldsymbol{k}'}$ defined previously (cf. (2.50)) and

$$u^{(n)}_{\boldsymbol{k},\boldsymbol{k}'} = \sum_{0\neq \boldsymbol{k}_1,\ldots,\boldsymbol{k}_{n-1}\in\mathbb{Z}^d} u_{\boldsymbol{k},\boldsymbol{k}_1} u_{\boldsymbol{k}_1,\boldsymbol{k}_2} \ldots u_{\boldsymbol{k}_{n-1},\boldsymbol{k}'}$$

which satisfies

$$\sum_{0\neq \boldsymbol{k}'\in\mathbb{Z}^d} |u^{(n)}_{\boldsymbol{k},\boldsymbol{k}'}|^2 = 1, \quad \forall\, n,\ \boldsymbol{k}. \tag{3.37}$$

Then

$$\begin{aligned}
\tilde{T}^{(n)}_{\epsilon,\alpha} \boldsymbol{e}_{\boldsymbol{k}_0} &= G_{\epsilon,\alpha} U_F^n G_{\epsilon,\alpha} \boldsymbol{e}_{\boldsymbol{k}_0} = G_{\epsilon,\alpha} U_F^n e^{-\epsilon|\boldsymbol{k}_0|^2} \boldsymbol{e}_{\boldsymbol{k}_0} = e^{-\epsilon|\boldsymbol{k}_0|^2} G_{\epsilon,\alpha} \sum_{0\neq \boldsymbol{k}_n\in\mathbb{Z}^d} u^{(n)}_{\boldsymbol{k}_0,\boldsymbol{k}_n} \boldsymbol{e}_{\boldsymbol{k}_n} \\
&= e^{-\epsilon(|\boldsymbol{k}_0|^2+|\boldsymbol{k}_n|^2)} \sum_{0\neq \boldsymbol{k}_n\in\mathbb{Z}^d} u^{(n)}_{\boldsymbol{k}_0,\boldsymbol{k}_n} \boldsymbol{e}_{\boldsymbol{k}_n}.
\end{aligned}$$

Now we define

$$S_n(\boldsymbol{k}_n) = \{\boldsymbol{k}_0 \in \mathbb{Z}^d \backslash \{0\} : u^{(n)}_{\boldsymbol{k}_0,\boldsymbol{k}_n} \neq 0\}.$$

Similar computations to these performed in Section 2.7 give the following general

upper bound for $\|\tilde{T}^{(n)}_{\epsilon,\alpha}\|$

$$\|\tilde{T}^{(n)}_{\epsilon,\alpha} f\|^2 \leq \sum_{0\neq \boldsymbol{k}_n\in\mathbb{Z}^d} \sum_{\boldsymbol{k}_0\in S_n(\boldsymbol{k}_n)} |\hat{f}(\boldsymbol{k}_0)|^2 \sum_{\boldsymbol{k}_0\in S_n(\boldsymbol{k}_n)} |u^{(n)}_{\boldsymbol{k}_0,\boldsymbol{k}_n}|^2. \tag{3.38}$$

For a toral automorphism one easily sees that

$$u^{(n)}_{\boldsymbol{k}_0,\boldsymbol{k}_n} = e^{-\epsilon(|\boldsymbol{k}_0|^\alpha + |A^n\boldsymbol{k}_0|^\alpha)}\delta_{\boldsymbol{k}_n, A^n\boldsymbol{k}_0} \tag{3.39}$$

and hence

$$\|\tilde{T}^{(n)}_{\epsilon,\alpha}\| = e^{-\epsilon \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}(|\boldsymbol{k}_0|^\alpha + |A^l\boldsymbol{k}_0|^\alpha)}.$$

The arithmetic minimization problem (3.5) corresponding to the dissipation time of $\hat{T}_{\epsilon,\alpha}$ now becomes

$$\min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d} \left(|\boldsymbol{k}|^\alpha + |A^n\boldsymbol{k}|^\alpha\right). \tag{3.40}$$

The key observation is that, by the same arguments as before, similar estimates to these given in (3.15) hold

$$C_1\left(\sum_{j=1}^d \hat{\lambda}_j^{2n}|L_j\boldsymbol{k}|^2\right)^\alpha \leq |\boldsymbol{k}|^\alpha + |A^n\boldsymbol{k}|^\alpha \leq C_2\sum_{j=1}^d \hat{\lambda}_j^{\alpha n}|L_j\boldsymbol{k}|^\alpha. \tag{3.41}$$

The remaining computations are the same verbatim so the dissipation time of $T_{\epsilon,\alpha}$ and the family $\tilde{T}^{(n)}_{\epsilon,\alpha}$ are equal asymptotically.

3.2.2 Affine toral maps

In this section we present a slight generalization of the main results concerning the asymptotics of the dissipation time of toral automorphisms. Namely, we consider here general affine transformations of the torus. The term *affine transformations* will be used here to refer to homeomorphisms of the torus with zero periodic but not necessary zero constant part (cf. Section 3.1.1) i.e. transformations of the form $\tilde{F} = F + \boldsymbol{c}$, where F is a toral automorphism and $\boldsymbol{c}$ is a constant shift vector.

We begin with a short discussion of the ergodicity of affine transforms.

The relation between ergodicity of a given affine transform $\tilde{F}$ and associated with it toral automorphism F is summarized in the following proposition (for the proof we refer to appendix B)

Proposition 3.18 *Let F be any toral automorphism. Then*
i) If F is ergodic then $\tilde{F}$ is also ergodic.
ii) If F is not ergodic then $\tilde{F}$ is ergodic iff 1 is the only root of unity in the spectrum of F and $\boldsymbol{c}\cdot\boldsymbol{k}\notin\mathbb{Z}^d$ for any integer eigenvector $\boldsymbol{k}$ of $F^\dagger$.

Proof. i) Assume F is ergodic and for some $\boldsymbol{c}$, $\tilde{F} = F + \boldsymbol{c}$ is not ergodic. Then there exists non-constant $f \in L_0^2(\mathbb{T}^d)$ satisfying $f = f \circ \tilde{F}$ or in the Fourier representation

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \hat{f}(\boldsymbol{k})\boldsymbol{e_k} = \sum_{\boldsymbol{k}\in\mathbb{Z}^d} e^{2\pi i A^{-1}\boldsymbol{k}\cdot\boldsymbol{c}}\hat{f}(A^{-1}\boldsymbol{k})\boldsymbol{e_k} \tag{3.42}$$

where $A = F^\dagger$. Comparing the absolute values of the coefficients we get

$$|\hat{f}(\boldsymbol{k})| = |\hat{f}(A^{-n}\boldsymbol{k})| \tag{3.43}$$

for any integer n and any $\boldsymbol{k}$. However, ergodicity of F implies that $A^{-n}\boldsymbol{k} \neq \boldsymbol{k}$ for all $\boldsymbol{k} \neq 0$, which contradicts our assumption that $f \in L_0^2(\mathbb{T}^d)$.
ii) We will use the following fact, which can be proved by simple application of rational canonical decomposition. For any $A \in SL(d,\mathbb{Z})$ the following conditions are equivalent
a) A possesses in its spectrum a root of unity not equal to one.
b) There exists nonzero $\boldsymbol{k} \in \mathbb{Z}^d$ and a positive integer n such that $\boldsymbol{k}+A\boldsymbol{k}+...+A^{n-1}\boldsymbol{k} = 0$.
Now assume that 1 is the only root of unity in spectrum of F (and hence of A) and $\boldsymbol{c}\cdot\boldsymbol{k} \notin \mathbb{Z}^d$ for any integer eigenvector $\boldsymbol{k}$ of A, and that both F and $\tilde{F}$ are not ergodic. The latter assumption implies the existence of a non-constant $f \in L_0^2(\mathbb{T}^d)$ satisfying equations (3.42) and (3.43). Relation (3.43) clearly implies that if $\hat{f}(\boldsymbol{k}) \neq 0$ then $A^n\boldsymbol{k} = \boldsymbol{k}$ for some n. Moreover, since 1 is the only root of unity in spectrum of A,

we have, in view of b) that $A\boldsymbol{k} = \boldsymbol{k}$. Thus the only possible non-constant invariant functions of $\tilde{F}$ are single Fourier modes $\boldsymbol{e_k}$ corresponding to integer eigenvectors of A. But if such a Fourier mode is invariant under $\tilde{F}$ then directly from (3.42) one concludes that $e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{c}} = 1$ or equivalently $\boldsymbol{k}\cdot\boldsymbol{c} \in \mathbb{Z}^d$, for some integer eigenvector of A. To prove the converse we assume that F is not ergodic and consider two cases:

Case 1. A possesses in its spectrum a root of unity not equal to one. In this case according to condition b) there exists nonzero $\boldsymbol{k} \in \mathbb{Z}^d$ and a positive integer n such that $\boldsymbol{k} + A\boldsymbol{k} + ... + A^{n-1}\boldsymbol{k} = 0$, which implies in particular that $A^n\boldsymbol{k} = \boldsymbol{k}$ and $A\boldsymbol{k} \neq \boldsymbol{k}$. Now we define the function

$$f = \boldsymbol{e_k} + e^{2\pi i\boldsymbol{k}\cdot\boldsymbol{c}}\boldsymbol{e}_{A\boldsymbol{k}} + ... + e^{2\pi i\left(\sum_{l=0}^{n-2} A^l\boldsymbol{k}\right)\cdot\boldsymbol{c}}\boldsymbol{e}_{A^{n-1}\boldsymbol{k}}$$

which clearly satisfies the condition $f = f \circ \tilde{F}$. This proves that $\tilde{F}$ is not ergodic.

Case 2. There exists integer eigenvector of A such that $\boldsymbol{k}\cdot\boldsymbol{c} \in \mathbb{Z}^d$. Then clearly for such $\boldsymbol{k}$, $f = e_{\boldsymbol{k}}$ is $\tilde{F}$-invariant and hence again $\tilde{F}$ is not ergodic. ■

We recall that $\mathbf{c} = (c_1, .., c_d)$ generates ergodic shift on the torus iff $1, c_1, .., c_d$ are linearly independent over rationals. Thus as a direct consequence of the above proposition we get

Corollary 3.19 *If F is not ergodic and 1 is the only root of unity in the spectrum of F then $\tilde{F}$ is ergodic for all vectors $\mathbf{c}$ generating ergodic shifts on the torus.*

Now we are in a position to state and prove the generalization of Theorem 3.6 from Section 3.1.2 to the case of affine transforms (the corresponding generalizations of Theorem 3.7 and Corollary 3.8 are straightforward)

Theorem 3.20 *Let $\tilde{F}$ be any affine transformation on the torus $\mathbb{T}^d$, F associated with $\tilde{F}$ toral automorphism and $T_{\epsilon,\alpha} = G_{\epsilon,\alpha}U_{\tilde{F}}$. Then*

i) $T_{\epsilon,\alpha}$ has simple dissipation time iff F is not ergodic.

ii) $T_{\epsilon,\alpha}$ has logarithmic dissipation time iff F is ergodic.

iii) If $T_{\epsilon,\alpha}$ has logarithmic dissipation time then the dissipation rate constant satisfies the following constraint

$$\frac{1}{\hat{h}(F)} \leq R_c \leq \frac{1}{\tilde{h}(F)},$$

where $\tilde{h}(F) \leq \hat{h}(F)$ is certain positive constant.

Remark 3.21 *The dissipation time of an affine transformation $\tilde{F}$ is determined by ergodic properties of its linear part F and hence not by ergodic properties of $\tilde{F}$ itself. In particular all ergodic affine transformations associated with nonergodic toral automorphisms (cf. Proposition 3.18) have simple dissipation time.*

Proof of Theorem 3.20 Specializing the general calculations of dissipation time presented in Section 2.7 to the case of affine transformations $\tilde{F} = F + \boldsymbol{c}$, with nonzero $\boldsymbol{c}$, one easily finds the following counterparts of formulas (2.55) and (2.56)

$$\begin{aligned} u_{\boldsymbol{k},\boldsymbol{k}'} &= e^{2\pi i \boldsymbol{k}\cdot\boldsymbol{c}} \delta_{A\boldsymbol{k},\boldsymbol{k}'}, \\ \mathcal{U}_n(\boldsymbol{k}_0, \boldsymbol{k}_n) &= e^{2\pi i \left(\sum_{l=0}^{n-1} A^l \boldsymbol{k}\right)\cdot\boldsymbol{c}} e^{-\epsilon^\alpha \sum_{l=1}^{n} |A^l \boldsymbol{k}|^\alpha} \delta_{A^n \boldsymbol{k}_0, \boldsymbol{k}_n}. \end{aligned}$$

Now, in order to determine the dissipation time of $T_{\epsilon,\alpha} = G_{\epsilon,\alpha} U_{\tilde{F}}$ one has to determine the asymptotics of $\|T^n_{\epsilon,\alpha}\|$ as n goes to infinity. According to the above formulas and formulas (2.54) and (3.4) from Sections 2.7 and 3.1.3 the value of $\|T^n_{\epsilon,\alpha}\|$ does not depend on $\boldsymbol{c}$, which reduces the proof to the case we already considered i.e. $\boldsymbol{c} = 0$. ■

3.2.3 Degenerate noise

In this section we compute the dissipation time for non-strictly contracting generalizations of α-stable transition operators. Instead of considering standard α-stable kernels of the form (2.15) one can allow for some degree of degeneracy of noise in

chosen directions by introducing the following family of noise kernels

$$g_{\epsilon,\alpha,B}(\boldsymbol{x}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^d} e^{-|\epsilon B\boldsymbol{k}|^\alpha} \boldsymbol{e}_{\boldsymbol{k}}(\boldsymbol{x}), \tag{3.44}$$

Where B denotes any $d \times d$ matrix with $\det B = 0$.

We denote by $G_{\epsilon,\alpha,B}$ the noise operator associated with $g_{\epsilon,\alpha,B}$. The degeneracy of B immediately implies that $\|G_{\epsilon,\alpha,B}\| = 1$ and hence the general considerations of sections 1 and 2 do not apply here. The answer to the question whether or not the dissipation time is finite depends on the choice of matrix B.

For simplicity we concentrate on the case when B is diagonalizable.

We call the eigenvector of B nondegenerate if it corresponds to nonzero eigenvalue.

Theorem 3.22 *Let F be any toral automorphism and $T_{\epsilon,\alpha,B} = G_{\epsilon,\alpha,B}U_F$. Assume that B is diagonalizable. Then*

i) If all nondegenerate eigenvectors of B^* lie in one proper invariant subspace of F then dissipation does not take place i.e. $\tau_c = \infty$.

ii) Otherwise the following statements hold.

a) $T_{\epsilon,\alpha}$ has simple dissipation time iff F is not ergodic.

b) $T_{\epsilon,\alpha}$ has logarithmic dissipation time iff F is ergodic.

c) If $T_{\epsilon,\alpha,B}$ has logarithmic dissipation time then the dissipation rate constant satisfies the following bounds

$$\frac{1}{\hat{h}(F)} \le R_c \le \frac{1}{\tilde{h}(F)},$$

with some constant $\tilde{h}(F) \le \hat{h}(F)$. The equality is achieved for all diagonalizable automorphisms F.

Proof.

We continue to use the convention $A = F^\dagger$. The general formula derived previously

for $\|T^n_{\epsilon,\alpha}\|$ (see (3.4)), will now take the form

$$\|T^n_{\epsilon,\alpha,B}\| = \sup_{0\neq\boldsymbol{k}\in\mathbb{Z}^d} e^{-\epsilon^\alpha \sum_{l=1}^n |BA^l\boldsymbol{k}|^\alpha} = \exp -\epsilon^\alpha \inf_{0\neq\boldsymbol{k}\in\mathbb{Z}^d} \sum_{l=1}^n |BA^l\boldsymbol{k}|^\alpha. \tag{3.45}$$

Thus we need to estimate

$$\inf_{0\neq\boldsymbol{k}\in\mathbb{Z}^d} \sum_{l=1}^n |BA^l\boldsymbol{k}|^\alpha.$$

To this end we denote by μ_j ($j = 1, ..., d$) the eigenvalues of B and we construct a basis (of $\mathbb{C}^d$) $\{\boldsymbol{v}_j\}_{j=1,\dots,d}$ composed of normalized eigenvectors corresponding to eigenvalues μ_j. We denote by ${P_j}_{j=1,\dots,d}$ the set of eigen-projections on $\boldsymbol{v}_j$, and by L_j the set of corresponding linear forms, given by the eigenvectors $\boldsymbol{u}_j$ of $B^\dagger$, which are of course co-orthogonal to $\boldsymbol{v}_j$, i.e. $\langle \boldsymbol{u}_i, \boldsymbol{v}_j\rangle = 0$ for $i \neq j$. We have

$$\boldsymbol{x} = \sum_{j=1}^d P_j\boldsymbol{x} = \sum_{j=1}^d (L_j\boldsymbol{x})\boldsymbol{v}_j = \sum_{j=1}^d \langle \boldsymbol{x}, \boldsymbol{u}_j\rangle \boldsymbol{v}_j, \quad \forall \boldsymbol{x} \in \mathbb{R}^d.$$

In subsequent computations the symbols C_1, C_2 denote some absolute constants values of which are subject to change during calculations.

We consider two cases.

i) All nondegenerate eigenvectors of $B^\dagger$ lie in one proper subspace of F. We have the following estimates

$$\begin{aligned} |BA^l\boldsymbol{k}|^2 &\geq C_1 \sum_{j=1}^d |P_jBA^l\boldsymbol{k}|^2 = C_1 \sum_{j=1}^d |\mu_j|^2 |P_jA^l\boldsymbol{k}|^2 \\ &= C_1 \sum_{j=1}^d |\mu_j|^2 |\langle A^l\boldsymbol{k}, \boldsymbol{u}_j\rangle|^2 = C_1 \sum_{j=1}^d |\mu_j|^2 |\langle \boldsymbol{k}, F^l\boldsymbol{u}_j\rangle|^2 \end{aligned}$$

and

$$\begin{aligned} |BA^l\boldsymbol{k}|^2 &\leq C_2 \sum_{j=1}^d |P_jBA^l\boldsymbol{k}|^2 = C_2 \sum_{j=1}^d |\mu_j|^2 |P_jA^l\boldsymbol{k}|^2 \\ &= C_2 \sum_{j=1}^d |\mu_j|^2 |\langle A^l\boldsymbol{k}, \boldsymbol{u}_j\rangle|^2 = C_2 \sum_{j=1}^d |\mu_j|^2 |\langle \boldsymbol{k}, F^l\boldsymbol{u}_j\rangle|^2 \end{aligned}$$

Since at least one of μ_j is zero and all nondegenerate vectors $\boldsymbol{u}_j$ lie in a proper invariant subspace of F, one easily sees that for each fixed n

$$\inf_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|BA^l\boldsymbol{k}|^\alpha = \inf_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}\sum_{j=1}^{d}|\mu_j|^\alpha|\langle\boldsymbol{k},F^l\boldsymbol{u}_j\rangle|^\alpha = 0.$$

ii) In this case we have the following upper bound

$$\inf_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|BA^l\boldsymbol{k}|^\alpha \leq \|B\|^\alpha \inf_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|A^l\boldsymbol{k}|^\alpha = \|B\|^\alpha \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|A^l\boldsymbol{k}|^\alpha. \tag{3.46}$$

In order to provide an appropriate lower bound we note that the set of vectors $\{F^h\boldsymbol{u}_j\}$, where $1\leq h\leq d$ and j runs through the indices of all nondegenerate eigenvectors of B, spans the whole space (otherwise all nondegenerate $\boldsymbol{u}_j$ would lie in one proper invariant subspace of F). We denote by $\{F^{h_i}\boldsymbol{u}_{j_i}\}$ $(1\leq i\leq d)$ a basis extracted from the above set. We can define now a new norm $|\cdot|_{\boldsymbol{u}}$ on $\mathbb{R}^d$ by

$$|\boldsymbol{x}|_{\boldsymbol{u}}^2 = \sum_{i=1}^{d}|\langle\boldsymbol{x},F^{h_i}\boldsymbol{u}_{j_i}\rangle|^2$$

and compute

$$\begin{aligned}
\sum_{l=1}^{dn}|BA^l\boldsymbol{k}|^\alpha &= \sum_{l=0}^{n-1}\sum_{h=1}^{d}|BA^{dl+h}\boldsymbol{k}|^\alpha \geq \sum_{l=0}^{n-1}\sum_{h=1}^{d}C_1\sum_{j=1}^{d}|P_jBA^{dl+h}\boldsymbol{k}|^\alpha \\
&= C_1\sum_{l=0}^{n-1}\sum_{h=1}^{d}\sum_{j=1}^{d}|\mu_j|^\alpha|P_jA^{dl+h}\boldsymbol{k}|^\alpha \geq C_1\sum_{l=0}^{n-1}\sum_{i=1}^{d}|\langle A^{dl+h_i}\boldsymbol{k},\boldsymbol{u}_{j_i}\rangle|^\alpha \\
&= C_1\sum_{l=0}^{n-1}\sum_{i=1}^{d}|\langle A^{dl}\boldsymbol{k},F^{h_i}\boldsymbol{u}_{j_i}\rangle|^\alpha = C_1\sum_{l=0}^{n-1}|A^{dl}\boldsymbol{k}|_{\boldsymbol{u}}^\alpha
\end{aligned}$$

Using the equivalence between norms $|\cdot|$ and $|\cdot|_{\boldsymbol{u}}$ and combining (3.46) with the above estimate we get

$$C_1\min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=0}^{n-1}|A^{dl}\boldsymbol{k}|^\alpha \leq \inf_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{dn}|BA^l\boldsymbol{k}|^\alpha \leq \|B\|^\alpha\min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{dn}|A^l\boldsymbol{k}|^\alpha.$$

This together with the obvious fact that $\hat{h}(A^d) = d\hat{h}(A)$ and the general estimate (3.6) reduces the proof back to the nondegenerate case considered in the previous section. ■

3.2.4 Time scales in kinematic dynamo

In this section we briefly discuss the connection between the dissipation time and some characteristic time scales associated with kinematic dynamo, which concerns the generation of electromagnetic fields by mechanical motion. For a general setup and discussion we refer the reader to [30] and [72] and references therein. Here we restrict ourselves only to necessary definitions.

Let $\boldsymbol{B} \in L_0^2(\mathbb{T}^d, \mathbb{R}^d)$ denote periodic, zero mean and divergence free magnetic field and let F be the time-1 map associated with the fluid velocity. We define the push-forward map

$$F_*\boldsymbol{B}(\boldsymbol{x}) = dF(F^{-1}(\boldsymbol{x}))\boldsymbol{B}(F^{-1}(\boldsymbol{x})).$$

The noisy push-forward map $P_{\epsilon,\alpha}$ on $L_0^2(\mathbb{T}^d, \mathbb{R}^d)$ is then given by

$$P_{\epsilon,\alpha}\boldsymbol{B} := G_{\epsilon,\alpha}F_*\boldsymbol{B}, \tag{3.47}$$

where the convolution (the action of $G_{\epsilon,\alpha}$) is applied component-wise.

It is said that the kinematic dynamo action (positive dynamo effect) occurs if the dynamo growth rate is positive i.e.

$$R_{dyn} = \lim_{n\to\infty} \frac{1}{n} \ln \|P^n_{\epsilon,\alpha}\| > 0.$$

Moreover if

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \ln \|P^n_{\epsilon,\alpha}\| > 0,$$

then the dynamo action is said to be fast; otherwise it is slow. The anti-dynamo action takes place if

$$\lim_{n\to\infty} \frac{1}{n} \ln \|P^n_{\epsilon,\alpha}\| < 0.$$

Now we introduce the *threshold time* scale as

$$n_{th} = \max\{n : \|P^n_{\epsilon,\alpha}\| > e \text{ such that } \|P^{n-1}_{\epsilon,\alpha}\| \text{ or } \|P^{n+1}_{\epsilon,\alpha}\| \leq e\}.$$

The threshold time $n_{th}(\epsilon)$ is of order $O(1)$ as $\epsilon \to 0$ for all fast kinematic dynamo systems. In the case of anti-dynamo action, $n_{th}(\epsilon)$ captures the longest time scale on which the generation of the magnetic field still takes place. Finally $n_{th}(\epsilon)$ is not defined for systems with do not exhibit any growth of magnetic field throughout the evolution. In the case of anti-dynamo we consider the time scale on which the generation of the magnetic field achieves its maximal value

$$n_p = \min\{n : \|P^n_{\epsilon,\alpha}\| = \sup_m \|P^m_{\epsilon,\alpha}\|\}.$$

which is called the *peak time* of the anti-dynamo action.

Our next theorem establishes the relation between n_p, n_{th} and τ_c for toral automorphisms. Thus $dF = F$ and

$$P_{\epsilon,\alpha}\boldsymbol{B} = g_{\epsilon,\alpha} * F(\boldsymbol{B} \circ F^{-1}).$$

Theorem 3.23 *Let F be any toral automorphism. Then*

i) If F is nonergodic and has positive entropy then for all $0 < \epsilon < R_c \ln \rho_F$ the fast dynamo action takes place with dynamo growth rate satisfying

$$R_{dyn} = \ln \rho_F - \epsilon R_c^{-1} \stackrel{\epsilon\to 0}{\longrightarrow} \ln \rho_F > 0,$$

where ρ_F denotes the spectral radius of F. The threshold time n_{th} is of order $O(1)$ and if F is diagonalizable then $n_{th} \approx [R^{-1}_{dyn}]$.

ii) If F is nonergodic and has zero entropy then anti-dynamo action occurs and for nondiagonalizable F,

$$n_p \sim \frac{n_{th}}{\ln(n_{th})} \sim \tau_c \approx R_c \frac{1}{\epsilon}.$$

Moreover there exists a constant $0 < \gamma \leq d$ such that $\|P^{n_p}_{\epsilon,\alpha}\| \sim (1/\epsilon)^\gamma$. If F is diagonalizable then $\|P^n_{\epsilon,\alpha}\|$ is strictly decreasing (in n) and, hence, $n_p = 0$ and n_{th} is not defined.

iii) If F is ergodic then anti-dynamo action occurs and $n_p \approx \tau_c$. In particular if F is diagonalizable then

$$\begin{aligned} n_p &\approx n_{th} - R_c \ln(n_{th}) \approx R_c \ln(1/\epsilon) \\ &= \frac{1}{\alpha \hat{h}(F)} \ln(1/\epsilon) \end{aligned}$$

and

$$\|P_{\epsilon,\alpha}^{n_p}\| \sim (1/\epsilon)^{\frac{\ln \rho_F}{\alpha \hat{h}(F)}}. \tag{3.48}$$

We see that even in the case of anti-dynamo action the magnetic field can still grow to relatively large magnitude when the noise is small (power-law in $1/\epsilon$).

Proof. Representing the initial magnetic field $\boldsymbol{B} = (\boldsymbol{b}_1, ..., \boldsymbol{b}_d)$ in Fourier basis

$$\boldsymbol{B} = \sum_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} \hat{\boldsymbol{B}}(\boldsymbol{k}) e_{\boldsymbol{k}}$$

one obtains

$$P_{\epsilon,\alpha} \boldsymbol{B} = \sum_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} F \hat{\boldsymbol{B}}(\boldsymbol{k}) e^{-\epsilon^\alpha |A\boldsymbol{k}|^\alpha} e_{A\boldsymbol{k}},$$

where we set $A = (F^{-1})^\dagger$. After n iterations

$$P_{\epsilon,\alpha}^n \boldsymbol{B} = \sum_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} F^n \hat{\boldsymbol{B}}(\boldsymbol{k}) e^{-\epsilon^\alpha \sum_{l=1}^n |A^l \boldsymbol{k}|^\alpha} e_{A^n \boldsymbol{k}}.$$

Thus

$$\begin{aligned} \|P_{\epsilon,\alpha}^n \boldsymbol{B}\|^2 &\leq \sum_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} |F^n \hat{\boldsymbol{B}}(\boldsymbol{k})|^2 e^{-2\epsilon^\alpha \sum_{l=1}^n |A^l \boldsymbol{k}|^\alpha} \\ &\leq \max_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} e^{-2\epsilon^\alpha \sum_{l=1}^n |A^l \boldsymbol{k}|^\alpha} \sum_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} |F^n \hat{\boldsymbol{B}}(\boldsymbol{k})|^2 \\ &= e^{-2\epsilon^\alpha \min_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} \sum_{l=1}^n |A^l \boldsymbol{k}|^\alpha} |F^n \boldsymbol{B}|^2 \\ &= e^{-2\epsilon^\alpha \sum_{l=1}^n |A^l \boldsymbol{k}_n|^\alpha} |F^n \boldsymbol{B}|^2 \\ &\leq e^{-2\epsilon^\alpha \sum_{l=1}^n |A^l \boldsymbol{k}_n|^\alpha} \|F^n\|^2 |\boldsymbol{B}|^2, \end{aligned}$$

where $\boldsymbol{k}_n$ denotes a solution of the minimization problem

$$\min_{0\neq\boldsymbol{k}\in\mathbb{Z}^d}\sum_{l=1}^{n}|A^l\boldsymbol{k}|^\alpha.$$

The above calculation provides the following upper bound

$$\|P^n_{\epsilon,\alpha}\| \;\leq\; e^{-\epsilon^\alpha\sum_{l=1}^{n}|A^l\boldsymbol{k}_n|^\alpha}\|F^n\|.$$

On the other hand let $\boldsymbol{v}_n$ denote a unit vector satisfying $\|F^n\|=|F^n\boldsymbol{v}|$. One immediately sees that the above upper bound for $\|P^n_{\epsilon,\alpha}\|$ is achieved for magnetic field of the form $\boldsymbol{B}=\boldsymbol{v}_n e_{\boldsymbol{k}_n}$. Thus

$$\|P^n_{\epsilon,\alpha}\| = e^{-\epsilon^\alpha\sum_{l=1}^{n}|A^l\boldsymbol{k}_n|^\alpha}\|F^n\|. \tag{3.49}$$

Now we consider the cases mentioned in the statement of the theorem

i) Nonergodic, nonzero entropy case.

For any nonergodic map we have

$$\sum_{l=1}^{n}|A^l\boldsymbol{k}_n|^\alpha \approx R_c^{-1}n.$$

This implies the following asymptotics

$$\|P^n_{\epsilon,\alpha}\| \approx e^{-\epsilon^\alpha R_c^{-1}n}\|F^n\| \approx e^{(-\epsilon^\alpha R_c^{-1}+\ln\rho_F)n+c_1\ln n+c_2}, \tag{3.50}$$

where $c_1, c_2 \geq 0$ are constants (both equal 0 iff F is diagonalizable). Thus for $\epsilon < R_c \ln\rho_F$ we have

$$R_{dyn} = \ln\rho_F - \epsilon^\alpha R_c^{-1} \xrightarrow{\epsilon\to 0} \ln\rho_F > 0.$$

The threshold time is clearly of order $O(1)$ and in diagonalizable case can be written as

$$n_{th} \approx \frac{1}{\ln\rho_F - \epsilon^\alpha R_c^{-1}} \xrightarrow{\epsilon\to 0} \frac{1}{\ln\rho_F}.$$

ii) Nonergodic, zero entropy case.

In this case $\ln \rho_F = 0$. Thus if F is nondiagonalizable then (3.50) reads

$$\|P^n_{\epsilon,\alpha}\| \approx e^{-\epsilon^\alpha R_c^{-1} n} \|F^n\| \approx e^{-\epsilon^\alpha R_c^{-1} n + c_1 \ln n + c_2},$$

with $0 < c_1 \leq d$. This immediately yields

$$n_p \approx R_c \frac{c_1}{\epsilon}, \qquad \frac{n_{th}}{\ln(n_{th})} \sim \frac{1}{\epsilon}.$$

And moreover $\|P^{n_p}_{\epsilon,\alpha}\| \sim (1/\epsilon)^{c_1}$.

If F is diagonalizable then $\|F^n\| = 1$ and in this case $\|P^n_{\epsilon,\alpha}\| \approx e^{-\epsilon^\alpha R_c^{-1} n}$ which implies $n_p = 0$.

iii) If F is diagonalizable, then from (3.6) we know that for any $0 < \delta < 1$ and sufficiently large n

$$\lambda^n_{-\delta} = e^{(1-\delta)\alpha \hat{h}(A) n} \leq \min_{0 \neq \boldsymbol{k} \in \mathbb{Z}^d} \sum_{l=1}^{n} |A^l \boldsymbol{k}|^\alpha \leq e^{(1+\delta)\alpha \hat{h}(A) n} = \lambda^n_{+\delta}. \tag{3.51}$$

Thus for large n we have

$$\max_n e^{-\epsilon^\alpha \lambda^n_{+\delta}} \rho^n_F \leq \max_n \|P^n_{\epsilon,\alpha}\| \leq \max_n e^{-\epsilon^\alpha \lambda^n_{-\delta}} \rho^n_F.$$

We obtain the following constraints for n_p

$$\frac{1}{\ln \lambda_{+\delta}} \ln\left(\frac{\ln \rho_F}{\ln \lambda_{+\delta}}\right) + \frac{1}{\ln \lambda_{-\delta}} \ln\left(\frac{1}{\epsilon}\right) \leq n_p \leq \frac{1}{\ln \lambda_{-\delta}} \ln\left(\frac{\ln \rho_F}{\ln \lambda_{-\delta}}\right) + \frac{1}{\ln \lambda_{-\delta}} \ln\left(\frac{1}{\epsilon}\right).$$

This gives

$$\frac{1}{\ln \lambda_{+\delta}} \leq \lim_{\epsilon \to 0} \frac{n_p}{\ln(1/\epsilon)} \leq \frac{1}{\ln \lambda_{-\delta}}.$$

Now since $\lambda_{\pm\delta} \to e^{\alpha \hat{h}(F)}$ for $\delta \to 0$ the above estimation yields the following asymptotics

$$n_p \approx \frac{1}{\hat{h}(F)} \ln(1/\epsilon) \approx \tau_c, \qquad n_{th} - R_c \ln(n_{th}) \approx \tau_c.$$

Similar asymptotic estimates (except for the constant) hold for nondiagonalizable F.

■

3.3 Dissipation time of general C^3 Anosov maps

We recall that a diffeomorphism $F : \mathbb{T}^d \mapsto \mathbb{T}^d$ is called Anosov if it is uniformly hyperbolic: there exist constants $A > 0$ and $0 < \lambda_s < 1 < \lambda_u$ such that at each $\boldsymbol{x} \in \mathbb{T}^d$ the tangent space $T_{\boldsymbol{x}}\mathbb{T}^d$ admits the direct sum decomposition $T_{\boldsymbol{x}}\mathbb{T}^d = E^s_{\boldsymbol{x}} \oplus E^u_{\boldsymbol{x}}$ into stable and unstable subspaces such that for every $n \in \mathbb{N}$,

$$(D_{\boldsymbol{x}}F)(E^s_{\boldsymbol{x}}) = E^s_{F\boldsymbol{x}}, \qquad \|(D_{\boldsymbol{x}}F^n)_{|E^s_{\boldsymbol{x}}}\| \leq A\lambda_s^n;$$
$$(D_{\boldsymbol{x}}F)(E^u_{\boldsymbol{x}}) = E^u_{F\boldsymbol{x}}, \qquad \|(D_{\boldsymbol{x}}F^{-n})_{|E^u_{\boldsymbol{x}}}\| \leq A\lambda_u^{-n}.$$

These inequalities have obvious consequences on the expansion rates of F and F^{-1}, for instance they imply $\|DF\|^n_\infty \geq \|DF^n\|_\infty \geq A^{-1}\lambda_u^n$. As a consequence, the quantities of interest in Theorem 2.17, *i)* satisfy

$$\begin{aligned} \|DF\|_\infty \wedge \|DF^{-1}\|_\infty &\geq \lambda_u \wedge \lambda_s^{-1}, \\ \mu_F \wedge \mu_{F^{-1}} &\geq \lambda_u \wedge \lambda_s^{-1}. \end{aligned}$$

All these expansion rates are > 1, so both noisy and coarse-grained dissipation times admit logarithmic lower bounds as in Eq. (2.34).

To obtain upper bounds, we use the mixing properties of this dynamics. Exponential mixing has been proved for Anosov diffeomorphisms of regularity $C^{1+\eta}$ $(0 < \eta < 1)$ by Bowen [26], using symbolic dynamics; the exponential decay is then valid for Hölder observables in $C^{\eta'}$ for some $0 < \eta' < \eta$.

Because we are interested in the noisy dynamics as well, we also refer to a more recent work [21] concerning C^3 Anosov maps on $\mathbb{T}^d$, which bypasses symbolic dynamics. The authors construct an *ad hoc* invariant Banach space $\mathcal{B}$ of generalized functions on the phase space, such that the Perron-Frobenius operator is quasicompact on this space. The difficulty compared to the case of expanding maps, is that the space $\mathcal{B}$ explicitly depends on the (un)stable foliations generated by the map F on $\mathbb{T}^d$. Vaguely speaking, the elements of $\mathcal{B}$ are "smooth" along this unstable direction $E^u_{\boldsymbol{x}}$, but can be singular

("dual of smooth") along the stable foliation $E^s_{\boldsymbol{x}}$. The space $\mathcal{B}$ is the completion of $C^1(\mathbb{T}^d)$ with respect to a weaker norm $\|\cdot\|_{\mathcal{B}}$ adapted to these foliations. The definition of that norm is given in terms of a parameter $0<\beta<1$, the choice of which depends on the *regularity* of the unstable foliation. In general, the latter is only Hölder continuous on the torus, with some exponent $0<\tau<2$ (note that the regularity of the foliations has little to do with the regularity of the map itself: a real-analytic map may have foliations being only Hölder). Then, one must take $\beta<\tau\wedge 1$, and the authors prove that the essential spectrum of P_F on $\mathcal{B}$ has a radius smaller than $r_1=\max(\lambda_u^{-1},\lambda_s^{\beta})$. This upper bound is sharper if β can be taken close to 1, that is, if the foliation is at least C^1. This is the case for smooth area-preserving Anosov maps on $\mathbb{T}^2$, for which the foliations have regularity $C^{2-\delta}$ for any $\delta>0$ [65]. The operator P_F may have isolated eigenvalues $1>|\lambda_i|>r_1$, corresponding to eigenstates in $\mathcal{B}$ which are genuine distributions, outside L^2_0. As opposed to the radius r_1, there is (to our knowledge) no simple general upper bound for largest resonance $|\lambda_1|$ in terms of $(\lambda_u,\ \lambda_s)$.

The space $C^1(\mathbb{T}^d)$ can be embedded continuously in both $\mathcal{B}$ and its dual $\mathcal{B}^*$, so that one can take $s=s_*=1$ in Eq. (3.54). Therefore, for any $1>\sigma>\max(|\lambda_1|,r_1)$, there is some constant $C>0$ such that for any $f,\,g\in C^1_0(\mathbb{T}^d)$,

$$\forall n>0,\qquad |C_{f,g}(n)|\le C\,\|g\|_{C^1}\,\|f\|_{C^1}\,\sigma^n. \tag{3.52}$$

In the proof of Theorem 2.20 (Step 3), for the case $s=s_*=1$ we only need to assume that the noise generating function g is C^1 with fast-decaying first derivatives. The fast decay implies that the first moment of g is finite (that is, one can take $\alpha\ge 1$).

The results of [21] also concern the noisy dynamics. If the unstable foliation has regularity $C^{1+\eta}$ with $\eta>0$ (for instance for any C^3 Anosov diffeomorphism on the 2-dimensional torus), and assuming the noise generating function $g\in C^2(\mathbb{R}^d)$ with *compact support*, the authors prove the strong spectral stability of the Perron-Frobenius

operator P_F on the space $\mathcal{B}$ defined with the parameter $\beta < \eta$. Therefore, the estimate (3.52) also applies to the noisy correlation function $C^{\epsilon}_{f,g}(n)$ as long as ϵ is small enough.

Using these estimates and applying Theorem 2.17 (Section 2.5) and Corollary 2.21 (Section 2.6) we obtain the following result regarding the dissipation time of C^3 Anosov maps on the torus:

Theorem 3.24 *Let F be a volume preserving C^3 Anosov diffeomorphisms on $\mathbb{T}^d$ and let g be a C^1 noise generating function with fast decay at infinity.*

I) Then there exist $\mu \geq \lambda_u \wedge \lambda_s^{-1}$, $0 < \sigma < 1$ and $\tilde{C} > 0$ such that the dissipation time of the coarse-grained dynamics satisfies

$$\frac{1}{\ln \mu} \ln(\epsilon^{-1}) - \tilde{C} \leq \tilde{\tau}_c \leq \frac{d+2}{|\ln \sigma|} \ln(\epsilon^{-1}) + \tilde{C},$$

II) If in addition F has $C^{1+\eta}$-regular foliations and $g \in C^2(\mathbb{R}^d)$ is compactly supported, then the dissipation time of the noisy evolution satisfies for some $C > 0$ and small enough ϵ:

$$\frac{1}{\ln \|DF\|_\infty} \ln(\epsilon^{-1}) - C \leq \tau_c \leq \frac{d+2}{|\ln \sigma|} \ln(\epsilon^{-1}) + C$$

3.4 Examples and general comments on other chaotic systems

3.4.1 Noiseless correlations

Let us first consider the correlation functions $C_{f,g}(n)$ for the noiseless dynamics. A common route to prove that a map F is mixing consists in studying the Perron-Frobenius operator P_F on some cleverly selected space $\mathcal{B}$ of densities (this may be a Banach or Fréchet space). The objective is to prove that the spectrum of P_F on $\mathcal{B}$ is of *quasicompact* type. More precisely, one shows that P_F admits 1 as simple eigenvalue

(with the constant eigenfunction), and that the radius of its essential spectrum is smaller than some $0 < r_1 < 1$. The spectrum outside the disk of radius r_1 is made of a finite number of isolated, finitely-degenerate eigenvalues $\{\lambda_i\}$ of moduli $r_1 < |\lambda_i| < 1$ (usually called Ruelle-Pollicott resonances). We order these eigenvalues according to their decreasing moduli, so that the largest modulus is $|\lambda_1|$. This spectral structure of P_F on $\mathcal{B}$ implies that for any $1 > \sigma > \max(|\lambda_1|, r_1)$ (if the largest resonance λ_1 is semisimple, one may take $\sigma = |\lambda_1|$), there is some constant $C = C(\sigma)$ such that for any $f \in \mathcal{B}_0$, $g \in \mathcal{B}_0^*$:

$$\begin{aligned} |C_{f,g}(n)| &= |\langle g, P_F^n f\rangle_{\mathcal{B}^*,\mathcal{B}}| \\ &\leq \|g\|_{\mathcal{B}^*} \|P_F^n\|_{\mathcal{B}_0} \|f\|_{\mathcal{B}} \\ &\leq C\, \|g\|_{\mathcal{B}^*}\, \|f\|_{\mathcal{B}}\, \sigma^n. \end{aligned} \tag{3.53}$$

The choice of the space $\mathcal{B}$ is crucial here: in general the spectrum of P_F on L^2_0 intersects the unit circle, so there is no chance to prove exponential decay within the L^2 framework. $\mathcal{B}$ can be a Banach space of bounded variation, Hölder or C^s functions which embeds continuously in L^2 (this is the choice made for F uniformly expanding, see Section 3.4.3); it may also be a space of generalized functions lying outside L^2 (case of Anosov maps, see Section 3.3).

In general, there exist Hölder exponents $0 \leq s_* \leq s$ such that C^s (resp. C^{s_*}) embeds continuously in $\mathcal{B}$ (resp. its dual $\mathcal{B}^*$). As a result, the upper bound (3.53) induces that for any $f \in C^s_0$, $g \in C^{s_*}_0$ and any $n > 0$,

$$|C_{f,g}(n)| \leq C\, \|g\|_{C^{s_*}}\, \|f\|_{C^s}\, \sigma^n. \tag{3.54}$$

This is the form of upper bound we used in Theorem 2.20. When the radius σ is given by a resonance λ_1, it can often not be decreased when one takes observables of higher regularity. In this case, it is preferable to take in the above estimate the weakest norms possible, that is take s and s_* as small as possible (to obtain better upper bounds in Corollary 2.21).

This strategy of proof has been applied to several types of maps [12, 13], including the (noninvertible) expanding maps and the Anosov or Axiom-A diffeomorphisms on a compact manifold. We will give some details and examples of these two types of maps in the next subsections. Exponential decay of correlations has also been proved (using various methods) for piecewise expanding maps on the interval, some nonuniformly hyperbolic/expanding maps, like "good" logistic maps on the interval, or "good" Hénon maps; some expanding or hyperbolic maps with singularities. In those cases, the decay rate may have no (obvious) spectral interpretation, as opposed to the formalism described above [13].

Different types of decay, like the stretched exponential $C_{f,g}(n) \leq K r^{n^\xi}$ with $0 < r < 1$ and $0 < \xi < 1$ have been proved for some Poincaré maps of hyperbolic flows and some random nonuniformly hyperbolic systems; yet, it seems that the bound may not be optimal in some of these cases, but rather due to the method of the proof. On the opposite, the polynomial decay $C_{f,g}(n) \lesssim n^{-\beta}$ was shown to be optimal for some "intermittent" systems, like a one-dimensional map expanding everywhere except at a fixed "neutral" point (such maps are sometimes called "almost expanding" or "almost hyperbolic").

Remark: In general, an expanding or Anosov map does not preserve the Lebesgue measure, so the first step before dealing with correlations is to precise the invariant measure with respect to which one wants to study the ergodic properties. In the "nice" cases, one can prove the existence and uniqueness of a "physical measure", which is ergodic for the map F, and then study the correlation functions with respect to this measure. The formalism of Perron-Frobenius operators applies also to this general case, the physical measure being related to the simple eigenstate of P_F (on the space $\mathcal{B}$) associated with the eigenvalue 1. As stated in the Introduction, in this work we only consider maps for which the physical measure is the Lebesgue measure.

3.4.2 Noisy correlations

There exist fewer results on the decay of correlations for stochastic perturbations of deterministic maps, like our noisy evolution T_ϵ. In general, one wants to prove some sort of *strong stochastic stability*, that is stability of the invariant measure, and of the rate of decay of the correlations in the limit when the noise parameter ϵ vanishes.

In the case of exponential mixing, this strong stochastic stability implies that for small enough ϵ, the upper bound (3.54) is "stable" when switching on the noise: for small enough $\epsilon > 0$ there exists a radius $\sigma_\epsilon \xrightarrow{\epsilon \to 0} \sigma$ such that for any $f \in C_0^s$, $g \in C_0^{s*}$,

$$\forall n > 0, \qquad |C^\epsilon_{f,g}(n)| \leq C\,\|g\|_{C^{s*}}\,\|f\|_{C^s}\,\sigma_\epsilon^n. \tag{3.55}$$

This stability has been proved for smooth uniformly expanding maps [14], some nonuniformly expanding or piecewise expanding maps (see the review in [13] or the book [12]). It has been shown also for uniformly hyperbolic (Anosov) maps on the 2-dimensional torus [21]. The proof uses perturbation theory: one shows that the isolated eigenvalues of $G_\epsilon \circ P_F$ on $\mathcal{B}$ of moduli $|\lambda_{i,\epsilon}| > r_1$ (and the associated eigenstates) vary continuously w.r.to ϵ when $\epsilon \to 0$. Therefore, one can choose a rate $\sigma_\epsilon > \max(|\lambda_{1,\epsilon}|, r_1)$ which is a continuous function of ϵ.

In the next two sections, we describe in some detail the exponential decay of correlations for smooth uniformly expanding maps and Anosov diffeomorphisms on the torus.

3.4.3 Smooth uniformly expanding maps

Let F be a C^{s+1} map on $\mathbb{T}^d$ (with $s \geq 0$). Assume that there exists $\lambda > 1$ such that for any $\boldsymbol{x} \in \mathbb{T}^d$ and any $\boldsymbol{v}$ in the tangent space $T_{\boldsymbol{x}}\mathbb{T}^d$, $\|DF(\boldsymbol{x})\boldsymbol{v}\| \geq \lambda\|\boldsymbol{v}\|$ (we assume that λ is the largest such constant). Such a map is called uniformly expanding. In general, it admits a unique absolutely continuous invariant probability measure; here we will restrict ourselves to maps for which this measure is the Lebesgue measure.

Ruelle [112] proved that the Perron-Frobenius operator P_F of such a map is quasi-compact on the space $C^s(\mathbb{T}^d)$, and that its essential spectrum is contained inside the disk of radius $r_1 = \lambda^{-s}$. In general, one has little information on the possible discrete spectrum outside this disk (upper bounds on the decay rate have been obtained in the case of an expanding map of regularity $C^{1+\eta}$ [12]). Strong stochastic stability for such maps was proved in [14], with a more general definition of the noise than the one we gave.

For all these cases, one can take $s_* = 0$, since the continuous functions are continuously embedded in any space $(C^s)^*$.

Case of a linear expanding map

We describe the simplest example possible for such a map, namely the angle-doubling map on $\mathbb{T}^1$ defined as $F(x) = 2x \bmod 1$. This map is real analytic, with uniform expansion rate $\lambda = 2$. Due to its linearity, the dynamics of this map (as well as its noisy version) is simple to express in the basis of Fourier modes $\boldsymbol{e}_k(x) = e^{2i\pi kx}$:

$$\begin{aligned} \forall k \in \mathbb{Z}, \quad U_F \boldsymbol{e}_k &= \boldsymbol{e}_{2k} \\ \Longrightarrow T_\epsilon \boldsymbol{e}_k &= \hat{g}(\epsilon k)\boldsymbol{e}_{2k} \\ \Longrightarrow T_\epsilon^n \boldsymbol{e}_k &= \Big[\prod_{j=1}^{n} \hat{g}(\epsilon 2^j k)\Big]\boldsymbol{e}_{2^n k} \end{aligned}$$

The computation is even simpler for the coarse-grained propagator:

$$\tilde{T}_\epsilon^{(n)} \boldsymbol{e}_k = \hat{g}(k)\hat{g}(2^n k)\boldsymbol{e}_{2^n k}.$$

To fix the ideas, we take for the noise generating function $\hat{g}(\xi) = e^{-|\xi|^\alpha}$ for some $0 < \alpha \le 2$. One easily checks that for any $n \ge 1$,

$$\begin{aligned} \|T_\epsilon^n\| &= \|T_\epsilon^n \boldsymbol{e}_1\| = \exp\Big\{ -\epsilon^\alpha \frac{2^{n\alpha}-1}{1-2^{-\alpha}} \Big\}, \\ \|\tilde{T}_\epsilon^{(n)}\| &= \exp\{-\epsilon^\alpha (2^{n\alpha}+1)\}. \end{aligned}$$

For any $\epsilon > 0$, these decays are super-exponential: the spectrum of T_ϵ on L_0^2 is reduced to $\{0\}$ for any $\epsilon > 0$ (the spectrum of U_F is the full unit disk). From there, we get explicit expressions for both dissipation times:

$$\tau_c = \frac{1}{\ln 2}\ln(\epsilon^{-1}) + \mathcal{O}(1), \qquad \tilde{\tau}_c = \frac{1}{\ln 2}\ln(\epsilon^{-1}) + \mathcal{O}(1). \tag{3.56}$$

For this linear map, $2 = \|DF\|_\infty = \mu_F$, so this estimate is in agreement with the lower bounds (2.34), the latter being sharp if $\alpha \in [1, 2]$. On the other hand, $\ln 2$ is also equal with the Kolmogorov-Sinai (K-S) entropy $h(F)$ of F. Therefore, for this linear map the dissipation rate constant exists, and is equal to $\frac{1}{h(F)}$.

To compare these exact asymptotics with the upper bounds of Corollary 2.21, we estimate the correlation functions $C_{f,g}(n)$ on the spaces $C^s(\mathbb{T}^1)$. We give below a short proof in the case $s > \frac{1}{2}$. We will use the following Fourier estimates [131]:

$$\exists C > 0, \quad , \forall f \in C_0^s(\mathbb{T}^1), \quad \forall k \neq 0, \qquad |\hat{f}(k)| \leq C\frac{\|f\|_{C^s}}{|k|^s}.$$

Therefore, writing the correlation function as a Fourier series, we get:

$$\begin{aligned} \|P_F^n f\|^2 &= \sum_{0\neq k\in\mathbb{Z}} |\hat{f}(2^n k)|^2 \leq \sum_{0\neq k\in\mathbb{Z}} \left(C\frac{\|f\|_{C^s}}{|2^n k|^s}\right)^2 \\ \Longrightarrow \|P_F^n f\| &\leq C' \frac{\|f\|_{C^s}}{(2^s)^n}. \end{aligned} \tag{3.57}$$

This estimate yields a decay of the correlation function as in Eq. (3.54), with a rate $\sigma = 2^{-s}$ and $s_* = 0$. One can check that this rate is sharp for functions in C^s: indeed, any $z \in \mathbb{C}$, $|z| < 2^{-s}$ is an eigenvalue of P_F on that space. Applying the Corollary 2.21,*I ii)*, we get that for any $s \geq 0$, there exists a constant $\tilde{c}$ such that

$$\tilde{\tau}_c \leq \frac{1+s}{s\ln 2}\ln(\epsilon^{-1}) + \tilde{c} \tag{3.58}$$

for sufficiently small ϵ. The exact dissipation rate constant $1/\ln 2$ is recovered only for large s.

A straightforward computation shows that the estimate (3.57) also holds if one replaces P_F by $P_F \circ G_\epsilon$; hence the noisy correlation function dynamics satisfies the same

uniform upper bound as the noiseless one, with the decay rate $\sigma_\epsilon = 2^{-s}$. As a result, the upper bound on τ_c given by Corollary 2.21,*II ii)* is the same as in Eq. (3.58).

3.5 Technical proofs

Proof of Proposition 3.13.

For the purposes of the proof we use the following abbreviation

- $PRS(\mathbb{R}^d)$ - proper rational subspace of $\mathbb{R}^d$.
- $PIS(A, \mathbb{R}^d)$ - proper A-invariant subspace of $\mathbb{R}^d$.
- $PRIS(A, \mathbb{R}^d)$ - proper rational A-invariant subspace of $\mathbb{R}^d$.

a) $\Rightarrow$ **b)** . Suppose there exists $PRIS(A, \mathbb{R}^d)$ S_1. Let A_1 be a matrix representing invariant rational block associated with S_1. Then A_1 is rational matrix and its characteristic polynomial P_1 belongs to $\mathbb{Q}[x]$. Let P denote the characteristic polynomial of A. Then $P = P_1P_2$ and since both $P, P_1 \in \mathbb{Q}[x]$ then also $P_2 \in \mathbb{Q}[x]$, which means P and hence A is not irreducible.

b) $\Rightarrow$ **c)** . Assume there exists $PRS(\mathbb{R}^d)$ S contained in $PIS(A, \mathbb{R}^d)$ V. Take any rational vector $\boldsymbol{q} \in S$ and let $d_0 = dimV$ then the set $\{\boldsymbol{q}, A\boldsymbol{q}, .., A^{d_0-1}\boldsymbol{q}\}$ spans $PRIS(A, \mathbb{R}^d)$.

c) $\Rightarrow$ **d)** . Assume that for given $\boldsymbol{q}$ and an arithmetic sequence $n_1, ..., n_d$, the set $S = \{A^{n_1}\boldsymbol{q}, A^{n_2}\boldsymbol{q}, ..., A^{n_d}\boldsymbol{q}\}$ does not form a basis. Since for some fixed integer r, $n_l = n_1 + (l-1)r$, we have $A^{n_l}\boldsymbol{q} = (A^r)^{l-1}A^{n_1}\boldsymbol{q} = (A^r)^{l-1}\hat{\boldsymbol{q}}$, where $\hat{\boldsymbol{q}} = A^{n_1}\boldsymbol{q}$. Now consider the biggest subset $S_0 = \{\hat{\boldsymbol{q}}, A^r\hat{\boldsymbol{q}}, (A^r)^2\hat{\boldsymbol{q}}, ..., (A^r)^{d_0-1}\hat{\boldsymbol{q}}\}$ such that $d_0 < d$ and S_0 is linearly independent. Obviously S_0 spans a $PRIS(A^r)$ which is also a $PRIS(A)$.

d) ⇒ a) . Suppose that characteristic polynomial P of A is not irreducible in $\mathbb{Q}[x]$. Then $P = P_1P_2$, with $P_1, P_2 \in \mathbb{Q}[x]$. From the Cayley-Hamilton theorem we get that $0 = P(A) = P_1(A)P_2(A)$. Hence for any nonzero rational vector $\boldsymbol{q}$, either 1) $P_2(A)\boldsymbol{q} = 0$ or 2) $\hat{\boldsymbol{q}} := P_2(A)\boldsymbol{q} \neq 0$ and $P_1(A)\hat{\boldsymbol{q}} = 0$. Since $\max\{deg(P_1, P_2)\} < d$, there exists a nonzero rational vector $\tilde{\boldsymbol{q}}$ (namely $\boldsymbol{q}$ or $\hat{\boldsymbol{q}}$) such that the set of iterations $\{\tilde{\boldsymbol{q}}, A\tilde{\boldsymbol{q}}, A^2\tilde{\boldsymbol{q}}, ..., A^{d-1}\tilde{\boldsymbol{q}}\}$ does not form a basis of $\mathbb{R}^d$.

e) ⇒ f) . Assume there exist nonzero $\boldsymbol{q} \in \mathbb{Q}^d$ orthogonal to certain $PIS(A, \mathbb{R}^d)$ V. Then for any n and any $f \in V$, $\langle (A^\dagger)^n\boldsymbol{q}, f\rangle = \langle \boldsymbol{q}, A^n f\rangle = 0$ and hence $S = \{\boldsymbol{q}, A^\dagger\boldsymbol{q}, (A^\dagger)^2\boldsymbol{q}, ..., (A^\dagger)^{d-1}\boldsymbol{q}\}$, cannot form a basis, which in view of equivalence a) ⇔ d) implies reducibility of $A^\dagger$.

f) ⇒ g) . Suppose there exists $PIS(A, \mathbb{R}^d)$ V contained in certain $PRS(\mathbb{R}^d)$ S. Since S is rational, $S^\perp$ is also rational. Consider any rational vector $\boldsymbol{q} \in S^\perp$, then $\langle \boldsymbol{q}, f\rangle = 0$ for any $f \in V$.

g) ⇒ b) . If there exists $PRIS(\mathbb{R}^d)$, then this subspace is A-invariant and contained in $PRS(\mathbb{R}^d)$ i.e in itself.

Now since b) is equivalent to a) it is enough to establish the equivalence between a) and e) to complete the proof. But the latter equivalence is obvious in view of the fact that A and $A^\dagger$ have the same characteristic polynomial. ■

Proof of Proposition 3.4.

Suppose A is a toral automorphism of zero entropy. The latter property is equivalent to the fact that all the eigenvalues of A are of modulus 1. Let P_A be a characteristic polynomial of A. Consider any irreducible over $\mathbb{Z}$ factor P of polynomial P_A and construct a toral automorphism B such that its characteristic polynomial is equal to P . Obviously all the eigenvalues of B are also the eigenvalues of A, and each

eigenvalue of A can be found among eigenvalues of some matrix B of this type. Irreducibility of P implies irreducibility and hence diagonalizability of B.

Thus for any nonzero vector $\boldsymbol{k} \in \mathbb{Z}^d$ and any positive integer n the following estimate holds $|B^n\mathbf{k}| \leq |\mathbf{k}|$, which implies the existence (for each $\mathbf{k}$) of some integer r such that $B^r\mathbf{k} = \mathbf{k}$.

The latter shows that all the eigenvalues of B (and hence also of A) are roots of unity.

■

Proof of Proposition 3.15.

We first show that irreducible polynomials $P \in \mathbb{Q}[x]$ do not have repeated roots. Indeed suppose λ is a root of P of multiplicity greater that 1, then λ is also a root of a derivative polynomial $P' \in \mathbb{Q}[x]$. Since the minimal polynomial of λ must divide both P and P' and $deg(P') < deg(P)$ one immediately concludes that P is not irreducible. Now, suppose $A \in GL(d, \mathbb{Q})$ is completely decomposable over $\mathbb{Q}$ and let (3.9) be its block diagonal decomposition into irreducible blocks. Each P_{A_j}, as a characteristic polynomial of A_j, is irreducible over $\mathbb{Q}$ and hence does not possesses repeated roots, which implies diagonalizability of each A_j and hence of A. On the other hand if A is diagonalizable then its minimal polynomial does not possesses repeated roots, which implies that all characteristic polynomials associated with elementary divisors are (first powers of) irreducible polynomials. This implies irreducibility of each block in representation (3.9). ■

Proof of Proposition 3.16.

Let P_A be the characteristic polynomial of an irreducible matrix $A \in GL(d, \mathbb{Q})$. Since P_A is an irreducible element of $\mathbb{Q}[x]$ it does not possesses repeated roots (see the proof of Proposition 3.15). ■

Proof of Proposition 3.9

Combining formula 3.4 and Theorem 3.10 in case of α-stable noises we get that for any $\delta > 0$

$$\|T^n_{\epsilon,\alpha}\| \leq e^{-\epsilon^\alpha e^{(1-\delta)\alpha \hat{h}(F) n}}. \tag{3.59}$$

Using this estimate one immediately gets

$$C^\epsilon_{f,h}(n) = \langle \bar{f}, T^n_{\epsilon,\alpha} h\rangle \leq \|f\|\|h\|\|T^n_{\epsilon,\alpha}\| \leq \|f\|\|h\| e^{-\epsilon^\alpha \lambda^{\alpha n}}.$$

Now let $f = G_\epsilon f_0$ and $h = G_\epsilon g_0$. Since the estimate (3.59) holds also in coarse-grained version, we have for $\alpha = 2$

$$\begin{aligned} C_{f,h}(n) &= \langle \bar{f}, U^n_F h\rangle = \langle G_\epsilon \bar{f}_0, U^n_F G_\epsilon h_0\rangle = \langle \bar{f}_0, \tilde{T}^{(n)}_\epsilon h_0\rangle \\ &\leq \|f_0\|\|h_0\|\|\tilde{T}^{(n)}_\epsilon\| \leq \|f_0\|\|h_0\| e^{-\epsilon^2 \lambda^{2n}}. \qquad \blacksquare \end{aligned}$$

Part II

Quantum Mechanics

Chapter 4

Interludium

This chapter - different in its form from the rest of the work - is meant as an introduction and historical overview of the development of the idea of quantum mechanics on the torus. We put an emphasis on the semiclassical analysis and we select only most important contributions to the subject directly pertaining to the problems considered in this work. Hence, although a lot of other important topics are going to be omitted, this introduction should be helpful for someone who wants to enter the field or at least understand some of its results without prior familiarity with it and without an intention to study all subsidiary contributions and developments which have been introduced during the now almost 25-years long history of the subject.

In the late seventies and early eighties both mathematicians and physicists realized the importance of and started to discuss seriously the problem of chaoticity in quantum dynamical systems [31, 127].

By that time the analysis of chaotic behavior of classical discrete-time dynamical systems on compact phase spaces (represented usually by a torus) constituted one of the most developed and best understood branches of the general theory of dynamical systems [10, 65, 36]. In fact, even today, most of the well-known and deeply understood examples of fully chaotic systems are described in terms of toral

maps (ergodic toral automorphisms, Baker and sawtooth maps, angle doubling maps, general Bernoulli shifts etc.).

Despite some controversy concerning physical interpretation (periodicity in momentum), from a mathematical point of view it became clear that chaotic toral maps should provide a convenient testing ground for a newly born field of study, now usually referred to as 'quantum chaos'. The quantization on the torus became at that point a necessity. As in the general case of quantization of any dynamical system the procedure is highly non-unique and from the very beginning different approaches had been taken and developed. Here we consider the two most popular ones, described respectively in finite and infinite dimensional settings. We start with the finite dimensional case.

In 1980 M.V. Berry and J.H. Hannay [63] made a first breakthrough in the area by introducing the notion of the finite dimensional quantum Hilbert space associated with a classical symplectic 2-torus. The dimension of the space in their model depends on the value of the Planck constant which itself is restricted to reciprocals of integers. The later condition results from the assumption, made by the authors, that quantum mechanical wave functions of any system on the torus should be periodic in both position and momentum representations. This assumption restricts the set of admissible pure states of the systems to periodic Dirac delta combs with equally spaced spikes. The distance between two neighboring delta functions in the comb is equal to the Planck constant (the smallest possible quantum resolution of the phase space) and since the whole comb is 1-periodic ('lives' on the torus) the constant can assume only inverse integer values $\hbar = 1/N$. Each delta comb can be identified with a set of its N complex amplitudes, i.e., the strengths associated to its 'delta spikes'. The corresponding quantum Hilbert space of all pure states is then N-dimensional and can be identified with $\mathbb{C}^N$.

This way the *kinematic* step of the quantization (i.e., description of the quantum

phase space) had been completed. The next step was to define the quantum *dynamics* on that space. Hannay and Berry chose to work with the well-known from the classical setting [10] discrete-time cat map dynamics. The assumption of the strict periodicity of their quantum phase space restricted in a considerable way the class of quantizable maps. The authors came to the conclusion that in order to be quantizable a cat map had to satisfy a so-called 'checkerboard condition', i.e., its matrix had to assume one of the following forms $\left[\begin{smallmatrix} e & o \\ o & e \end{smallmatrix}\right]$ or $\left[\begin{smallmatrix} o & e \\ e & o \end{smallmatrix}\right]$, where '$e$' and '$o$' denote respectively even and odd integer entries.

Under this assumption the authors constructed the quantum propagator ($N \times N$ unitary matrix) associated with a given map. One of the key points of choosing the cat map dynamics for quantization lay in the fact that the Hamiltonian of the classical map is in this case quadratic (see Section A.1 in Appendix A) and hence the semiclassical approximation to the Green function of the corresponding Schrödinger equation on $\mathbb{R}^2$ is exact [89]. This allowed the authors to define the quantum propagator on the torus as a 'projection' of the standard propagator on the plane. The projection, which in this case amounts to the discretization (applying the propagator to a delta comb) and periodization (wrapping the result around the torus) was performed formally by averaging of the kernel of the planar quantum propagator over all integer winding numbers associated with a classical path on the tours. This way the authors constructed a discrete and finite kernel ($N \times N$ matrix) representing the unitary quantum propagator of a toral map. The entries of the matrix were expressed in terms of oscillatory Gauss sums and the method introduced considerable number-theoretical complications as well as some difficult to resolve normalization issues. It thus became clear that except for some trivial cases the explicit formula for the matrix of the propagator was unattainable. Nevertheless, the authors were able to prove periodicity of such a quantum propagator, i.e., the existence of an integer $P(N)$ such that $U^{P(N)} = e^{i\phi(N)}\mathbb{1}$. This result yielded the following important information re-

garding the spectrum of U: each eigenvalue $e^{i\alpha_j}$ of U is constrained to take one of the possible $P(N)$ values specified in terms of their eigenangles

$$\alpha_j = \frac{2\pi j + \phi(N)}{P(N)}. \tag{4.1}$$

Not all of the values have to be assumed however, and on the other hand, some eigenvalues may be (and in fact usually are) highly degenerate. Thus in order to determine the spectrum one needs to find the period P(N) and the multiplicities d_j associated with each α_j. Hannay and Berry conjectured (and supported the conjecture with numerical evidence) that for regular maps $P(N)$ would be either bounded (elliptic case) or grow linearly with N (parabolic case) and that it would grow 'in average' as N but erratically in the chaotic (i.e., hyperbolic) case.

This line of research had been continued by Tabor [120] and further developed in 1991 by J.P. Keating [68, 69]. Using probabilistic number-theoretical approach supported by numerical computations Keating argued [68] that for chaotic cat maps $P(N)$ grows in fact slightly sublinearly (see below) and indeed highly erratically - the fluctuations themselves being of the order of N. More precisely, Keating studied the asymptotic behavior of the average order of $P(N)$ denoted by $\langle P(N)\rangle$ and defined as

$$\langle P(N)\rangle = \frac{1}{N}\sum_{n=1}^{N} P(n).$$

The reason to study $\langle P(N)\rangle$ instead of $P(N)$ is that as opposed to $P(N)$ the cumulant function $C(N) := \sum_{n=1}^{N} P(n)$ behaves very regularly and grows in N 'almost' like N^2. The normalized cumulant, i.e., the averaged order $\langle P(N)\rangle$ is then particularly suitable for asymptotic analysis. The result obtained by Keating states that for arbitrary small $\delta > 0$ and arbitrary big $\rho > 0$,

$$N^{1-\delta} \lesssim \langle P(N)\rangle \lesssim (\ln N)^{-\rho} N. \tag{4.2}$$

From this Keating concluded that in average $P(N)/N$ tends in fact to zero, although at a very slow rate.

As to the precise behavior of $P(N)$ itself (for large N) not much can be said because of its chaotic dependence on N. The size of the fluctuations of $P(N)$ had been estimated analytically by Kurlberg and Rudnick in [77, 78]. The result states that there exist constants c and C such that for every N

$$c\ln(N) < P(N) < CN\ln(\ln(N)). \tag{4.3}$$

It is worth pointing out that the bounds are sharp. In particular it has been proved recently in [54] that for every hyperbolic cat map there exists a sequence $\{N_k\}$ such that $P(N_k) \sim \ln(N_k)$. As will be described later in more details the existence of such sequences plays a crucial role in the semiclassical analysis of ergodic properties of quantum cat maps.

Once some knowledge about the properties of the quantum period function had been accumulated the next step was to study the degeneracies d_j of each eigenvalue to obtain the information about the spectrum of U.

It is easy to show (see [69]) that d_j can be represented in terms of the trace of the powers of the propagator U:

$$d_j = \frac{1}{P(N)} \sum_{k=1}^{P(N)} \mathrm{Tr}(U^k) e^{-k\alpha_j}. \tag{4.4}$$

In [69] Keating estimated the value of d_j for cat maps by deriving an exact version of the Gutzwiller's formula [62]

$$\mathrm{Tr}(U^k) = \frac{1}{\sqrt{R_k}} \sum_{m,n\in\mathcal{P}} \exp\left(\frac{i\pi N}{R_k}(c^k m^2 - b^k n^2 + 2(d^k - 1)mn)\right), \tag{4.5}$$

where $F = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, $R_k = \det(F^k - 1) = 2 - \mathrm{Tr}\, F^k$ and $\mathcal{P}$ denotes the fundamental parallelogram formed by the action of the planar linear map $F^k - I$ on the unit square. This parallelogram encompasses all the representatives of the periodic orbits of F of period k (all nonequivalent fixed points of F^k).

Thus just like in the case of the general Gutzwiller's formula, (4.5) relates the trace of U^k and hence in view of (4.4) and (4.1) the distribution of the eigenvalues

(the spectrum) of the quantum propagator U to its average over periodic orbits of the classical map (i.e., over the fixed points of F^k). It has however two important features which distinguish it from the general Gutzwiller's formula. First of all, the summation does not constitute merely a semiclassical approximation but gives the exact value of the trace for all (allowed on the torus) values of Planck constant $\hbar = 1/N$. Secondly, the formula is explicit. Thus in this particular case the most important feature of Gutzwiller's formula, i.e., the fact that it translates 'hard-to-compute' iterations of the quantum propagator into 'easy-to-compute' powers of the classical map, can be used to compute exactly the spectrum of U. For a fixed and small value of N this can be done numerically. For large N (which is important for semiclassical analysis) the number of periodic orbits proliferates exponentially and the numerical approach becomes impractical. But the formula remains useful for asymptotic analysis especially since the behavior of the periodic orbits of classical cat maps is now well understood (for further details in this direction we refer to [105] and [68]).

Having described the first stages of the development of the finite dimension quantization on the torus we now compare it with another approach - the infinite dimensional one, developed almost simultaneously but by different authors and for different reasons.

Just like the spectral approach to quantum chaoticity provided the main motivation for the introduction of the finite dimensional quantization on the torus, the infinite dimensional approach was initiated during the attempts to extend the notion of the entropy and K-property from classical ergodic theory to the quantum setting or more generally to the case of dynamical systems defined on noncommutative algebras.

Below we briefly discuss the role which the study of quantum entropy played in the development of the infinite dimensional noncommutative mechanics on the torus. We want to stress that the discussion is not meant to summarize the whole (much

longer) history of the notion of quantum entropy but to mention the developments relevant to quantization on the torus. In particular we will not have space here to discuss (but instead take for granted) von Neumann's definition of the entropy of the density matrix [97, 98], its extension to normal states on von Neumann algebras (obtained after the work by Araki [9]), Lieb's results on strong subadditivity and WYD conjecture [80, 81] etc. For a comprehensive overview of these notions and results we refer to [102].

In 1975 A. Connes and E. Størmer [35] introduced the notion of the entropy for automorphisms on certain noncommutative von Neumann algebras, namely on a hyperfinite type II_1 factor (see Definition 5.6). These algebras occur in the context of quantization on the torus. As was suggested by the authors, the main difficulty in generalizing the notion of the classical KS-entropy to non abelian setting seemed to lay in the fact that two finite dimensional subalgebras of the nonabelian von Neumann algebra need not generate a finite dimensional algebra (for a simple example see [94]), and hence there was no immediate analogue of the classical operation of the refinement $P \vee Q$ of two finite partitions P, Q of a phase space - a key notion used in the classical setting. We will see later in the discussion that an appropriate analogue in fact can be constructed by means of operational partition of unity but it leads to a different notion of quantum entropy [so-called ALF-entropy]. The solution the authors provided at that time was the following: instead of replacing the notion of the refinement of two partitions by its quantum analogue, the notion of its entropy $h_{KS}(P \vee Q)$ was replaced by a nonabelian counterpart $H(P, Q)$. The way the replacement can be made is, as one can expect, non-unique and different choices led to nonequivalent notions of quantum entropy. An interesting feature is that different but classically equivalent approaches can still yield nonequivalent quantum counterparts. Indeed, Connes and Størmer constructed a replacement for $h_{KS}(P \vee Q)$ based on its original definition, while independently but at almost the same time Emch [47] introduced

another replacement, based on a (classically equivalent) approach via conditional entropy. The resulting quantum notions did not coincide. In the original work by Connes and Størmer the noncommutativity of the algebra was 'overshadowed' to some extent by the requirement that the invariant state of the dynamics be tracial. In this case the state itself does not see the noncommutativity of the underlying algebraic structure. The generalization to nontracial states and to arbitrary C^*- and W^* algebras was achieved by Connes, Narnhofer and Thirring in 1987 [34] and resulted in the now well-known CNT entropy (in 1992 Sauvageot and Thouvenot introduced an alternative definition [116] and showed that for hyperfinite algebras all three, i.e., CS, CNT and 'ST' entropies coincide).

The main idea of the CNT construction is to introduce an abelian model of the noncommutative algebra and to transfer via a completely positive unital (CPU) map the invariant state and the dynamics from the nonabelian to the abelian setting. After appropriate corrections (taking into account the 'entropy defect' introduced by the CPU map) the entropy is computed in a classical manner.

After the definitions had been established the authors of the newly introduced notions started to look for examples of dynamical systems on which the notions could be tested and compared. As in the previous (finite dimensional) case the well known examples of classical chaotic toral maps once again came to the attention. At that time, however, it seemed obvious that an infinite dimensional model is absolutely necessary [that it is not exactly the case will become clear later in the discussion]. The main argument was that, just like in the classical case, in principle, the system can have nonzero entropy only if the generator (Koopman operator) of the dynamics possesses a continuous spectrum (cf. [36]). The latter requirement obviously calls for an infinite dimensional algebra of observables [this was the main reason why CS and CNT entropies were introduced in infinite dimensional settings].

Thus in 1991 Benatti, Narnhofer and Sewell [19] introduced a non-commutative

version of the Arnold cat map. Unlike in the case of the finite dimensional quantization, the authors didn't pay attention to the underlying physical space of pure states of the system and instead began the construction by introducing the appropriate noncomutative algebra of (quantum) observables. The most natural choice was the discrete (countably generated) Weyl-Heisenberg algebra spanned by the elements $W_{\boldsymbol{k}}$ (Weyl translations) indexed over the integral lattice $\mathbb{Z}^{2d}$ and satisfying the standard canonical commutation relations (CCR)

$$W_{\boldsymbol{k}} W_{\boldsymbol{m}} = e^{\pi i h \boldsymbol{k} \wedge \boldsymbol{m}} W_{\boldsymbol{k}+\boldsymbol{m}}, \tag{4.6}$$

$$W_{\boldsymbol{k}} W_{\boldsymbol{m}} = e^{2\pi i h \boldsymbol{k} \wedge \boldsymbol{m}} W_{\boldsymbol{m}} W_{\boldsymbol{k}}. \tag{4.7}$$

[The detailed technical description including the explanation of the seemingly non-standard placement for the Planck constant will be given in the next Chapter].

After introducing the cat map dynamics and an appropriate tracial state corresponding to the Lebesgue measure of a classical system the authors constructed the GNS representation, and hence in particular the Hilbert space $((l^2(\mathbb{Z}^2)))$ of states for the system. It turned out that the unitary map implementing the quantum automorphism on this space coincided with the Koopman operator of the classical map. Since the result turned out to be independent of the value of the Planck constant it became clear that the model would not be suitable for semiclassical considerations.

Nevertheless, the model still seemed adequate for the computations of CS-CNT entropies. Indeed, the above mentioned algebra is hyperfinite and either reduces to a tensor product of $L^\infty(\mathbb{T}^2)$ and a finite matrix algebra (if h is rational) or is irreducible and of type II_1 in irrational case (cf. Definition 5.6). In either case both CS and CNT entropies are well defined and coincide. In the rational case the CS entropy of a quantum cat map was shown [73] to coincide with a classical one. But the result did not contribute any new information as far as the quantum setting is concerned. Indeed, the cat map dynamics factorizes into a purely classical part acting on $L^\infty(\mathbb{T}^2)$

with its classical entropy and a finite dimensional quantum remainder with trivially vanishing quantum entropy.

In hope to obtain some nontrivial information one needed to look at 'irrational-h' case. It soon became clear, however, that in this case exceptionally strong properties of clustering and asymptotic abelianness [93, 91, 96] were necessary to secure the possibility of non-vanishing CNT entropy. Without entering into the details of the definitions, let us only recall that the notion of strong asymptotic abelianness was introduced in [18] and that it was basically shown in [8] that its lack implies zero CNT entropy. It was then found [92] that indeed, except for a possibly measure zero set of the values of the Planck constants, the CNT-entropy of the cat map is in fact zero. On top of that, in the course of the investigations, it turned out [95] that using the above mentioned property one can show that CNT-entropy fails to be additive w.r.t. the tensor products. Thus although CS-CNT entropies had been and are successfully applied to different dynamical systems another notion of noncommutative entropy was still needed.

In 1994 R. Alicki and M. Fannes [6] introduced a new approach to quantum dynamical entropy. Finite partitions used in the definition of the classical KS entropy had been replaced by the authors by (also finite) operational partitions of unity in corresponding quantum algebra. Such partitions could be evolved with a dynamics and composed among themselves yielding at each step finer but always finite new operational partitions. Using the standard notion of the von Neumann entropy of a quantum state and applying the procedure similar to the one known from the classical setting the authors constructed a new quantum dynamical entropy, now usually referred to as ALF entropy (in recognition of the connection of their idea to an earlier work by G. Lindblad [82]).

The definition was clearly compatible with a classical one in abelian case. Moreover it was shown [4] that, e.g., in the case of toral automorphisms in arbitrary di-

mensions the computations of the classical KS entropy using the operational approach could be significantly simplified in comparison to the traditional approach [126]. The entropy has also been successfully computed for quantized cat maps in the infinite dimensional setting [3] with the result identical to the classical one for all values of Planck constant (the result is consistent with the fact that the generator of the dynamics is independent of its value). The latter example indicated that the notion differs significantly from CS-CNT entropies. In fact there even exist examples of systems for which $H_{ALF} = \infty$ while $H_{CNT} = 0$ (see [8]). The two entropies differ also in the cases where the latter one is nonzero (e.g., for the shifts on quantum spin chains [6, 8]). Nevertheless, similarly as all previously mentioned quantum dynamical entropies, ALF entropy also vanishes for any fixed finite-dimensional quantum dynamical system - the phenomena known as *saturation*. The word *saturation* is a key notion here and it is worth to pause for a while and explain the source of the phenomenon in more detail.

In the classical setting KS entropy is defined as an infinite-time limit of the appropriately rescaled entropy production which occurs during the evolution of the system. Since the classical phase space can be refined into arbitrary small partitions, the rescaled entropy production converges to a well-defined limit and that limit is understood as the entropy of the system (in some, but not most interesting, cases the limit may be infinite).

In the finite dimensional quantum setting, with the phase-space resolution constrained by the value of the Planck constant, it is clear that after an initial stage of positive entropy production the growth of the entropy of the evolved partition comes to an end as further iterations do not introduce any new information. When this stage is achieved (i.e., the saturation takes place), the rescaling factor dominates the asymptotics and the infinite-time limit is zero.

The above analysis suggests the following solution. In order to get nontrivial

entropy results in the finite dimensional setting, instead of taking an infinite-time limit with a fixed Planck constant, one should consider simultaneous semiclassical and long time limits computing the entropy production for each value of positive but eventually vanishing Planck constant only up to certain 'saturation' time. If the limits are taken in this particular way and order, the infinite time limit should be nonzero and should provide truly valuable information about chaotic behavior of the underlying system. In particular, the longest time scale on which the above procedure recovers classical entropy can be thought of as a kind of "breaking" time, after which classical and quantum evolutions no longer agree. Some recent preprints [7, 16, 17] confirm the usefulness of the idea. In particular in [16] the authors compute semiclassical limit of both CNT and ALF entropy productions (on logarithmic in $\hbar$ time scales) for cat maps and prove that both notions lead to the same classical result. In particular the results remain in full agreement with our results (see Theorem 5.17) providing the base for the interpretation of the quantum dissipation rate constant as quantum dynamical entropy of the system (at least in 2-dimensional setting). It is also worthwhile to note at this point that the agreement holds exactly in the situation when the dissipation time does not exceeds the "breaking" time of the system.

The problem of "breaking" time is in some sense universal and emerges in all aspects of semiclassical analysis of quantum chaos. In order to understand it better we need to come back for a while to the description of the finite dimensional quantization procedure.

After initial works and promising results by Hannay, Berry, Tabor and Keating a need arose to clarify technical issues of finite dimensional quantization and in particular to generalize the scheme to allow quantization of at least all toral automorphisms (not only cat maps satisfying the 'checkerboard' condition) but eventually also Anosov maps and in general all canonical maps (symplectic diffeomorphisms) on the torus.

The first major breakthrough in this direction came in 1993 with a paper by Mirko Degli Esposti [43] in which the author suggested a representation-theoretical approach to the finite dimensional quantization (for more modern treatment see [44, 45]). Just like in the infinite dimensional case the key ingredient of the quantization was the Weyl representation of the discrete Heisenberg group, with the distinction that the representation was considered here over finite dimensional Hilbert space $L^2(\mathbb{T}, \mu)$, with purely atomic measure μ. The resulting *-algebras are indexed by rational values of h. Unlike the standard Weyl representation over the usual infinite dimensional quantum Hilbert space $L^2(\mathbb{R})$, where by the Stone-von Neumann theorem there is essentially unique irreducible infinite dimensional representation, for any fixed rational h there are infinitely many irreducible Weyl representations over finite dimensional space (in particular $L^2(\mathbb{T}, \mu)$). The representations are indexed by a parameter $\boldsymbol{\theta} \in \mathbb{T}^2$ and all share the same dimension N (if $h = p/q$ then N is the smallest integer such that Nh is an integer [61]). Using this “extra space” provided by the non-uniqueness of the infinite dimensional representations the author was able to remove the old checkerboard quantization condition and hence proved that for every cat map (i.e., 2-dim hyperbolic automorphism of the torus) there exists a finite dimensional Weyl representation on which the corresponding quantum Koopman operator acts as an (inner) *-automorphism (see Section 5.2.1). It was soon realized that the abstract Hilbert space $L^2(\mathbb{T}, \mu)$ can be considered as a generalization of the original Hannay and Berry space in a sense that strict periodicity of a wave function is replaced by a weaker condition of θ-quasiperiodicity defined as follows

$$\psi(q + m) = e^{2\pi i \theta_p m}\psi(q),$$

where m is an integer and $\theta_p \in \mathbb{T}$. An analogous condition is required for the momentum representation (with a corresponding Bloch angle $\theta_q \in \mathbb{T}$). For a given cat map $F = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ the appropriate quantization condition on the joint Bloch angle $\boldsymbol{\theta} = (\theta_q, \theta_p)$

has been established (see e.g. [43, 70])

$$\frac{N}{2}\begin{pmatrix} a\cdot b \\ c\cdot d \end{pmatrix} + F\boldsymbol{\theta} = \boldsymbol{\theta} \mod 1,$$

and after the work of Keating, Mezzadri and Robbins [70] it is usually referred to as 'quantum boundary condition'. In the same paper the authors generalized the quantization scheme to arbitrary canonical maps on the torus (some additional technical issues of the multiplicativity of the unitary propagators implementing the dynamics had been addressed later by Mezzadri, Kurlberg and Rudnick [90, 77], but since they do not pertain directly to our study, the results will not be described here).

In the meantime and afterward, different authors introduced finite dimensional quantization procedures for almost all other (e.g., discontinuous) well-known classical systems on the torus (see, e.g., [15, 114, 115] for quantization of the Baker map and [41] for quantization of the Sawtooth map).

Thus, after almost twenty years of development, the finite dimensional quantization on the torus had finally been established as a general and rigorous procedure and successfully applied to a large variety of toral maps.

In 1996 A. Bouzouina and S. De Bièvre [24] introduced an elegant and highly efficient mathematical formalism and notational setting in which the procedure could be described in a natural and transparent way. The authors started with a standard Weyl quantization on the usual Hilbert space $L^2(\mathbb{R})$, extended unitary Weyl translation operators (generators of the algebra) to the space of all tempered distributions $\mathcal{S}'(\mathbb{R})$ and then restricted their action to the space of quasiperiodic wave functions. The construction seems more natural then previous ones since instead of starting from an abstract representation theory, the appropriate algebra of quantum observables is derived directly from the traditional one. Thanks to this, any quantum observable corresponds in an natural way to its classical counterpart through its decomposition in the quantum Fourier basis formed by Weyl translation operators.

The method also provided a transparent link between the finite and the infinite dimensional settings (although the authors did not address this issue in their work directly). Indeed, if we denote by $\mathcal{A}$ the infinite dimensional algebra of the model introduced by Benatti, Narnhofer and Sewell [19] and restrict the values of the Planck constant to rationals then in the formalism introduced by Bouzouina and De Bièvre we simply have (cf. Section 5.1.5)

$$\mathcal{A} = \int_{\mathbb{T}^2}^{\oplus} \mathcal{A}_N(\boldsymbol{\theta}),$$

where $\mathcal{A}_N(\boldsymbol{\theta})$ denotes finite dimensional algebras of observables over the spaces of $\boldsymbol{\theta}$-quasiperiodic wave functions (and in Degli Esposti's formalism $\mathcal{A}_N(\boldsymbol{\theta})$ run through all the irreducible representations of the discrete Heisenberg group as $\boldsymbol{\theta}$ varies in $\mathbb{T}^2$). Moreover as will be shown in Sections 5.1 and 5.2 both finite and infinite dimensional quantizations can be constructed within the framework similar to the one introduced by Bouzouina and De Bièvre, provided one defines in an appropriate way the Hilbert spaces of pure states to which the action of the Weyl operators is to be restricted.

After clarifying quantization issues we can now return to the discussion of the breaking time. We did not define this notion precisely yet and in fact there is still no agreement in the literature on how it should be defined and even named. In an attempt to formalize this notion let us recall that according to Bohr's correspondence principle, the quantum evolution of any observable should approximate its classical evolution better and better as the Planck constant tends to zero. In mathematical literature the principle is expressed in terms of the so-called Egorov property ([42, 64, 109]), which states that for any smooth classical observable f and the classical evolution operator U, and for their corresponding quantizations $Op(f)$, $\mathcal{U}$ the following estimate holds

$$||\mathcal{U}^t Op(f) - Op(U^t(f))|| \leq C_{t,f}\hbar. \tag{4.8}$$

In the above estimate the constant in general depends in an essential way on both

t and f. The key point is that it is $\hbar$-independent and hence in the classical limit the quantum and the classical evolutions "commute". We want to stress that even in case of very reasonable quantization procedures the Egorov property cannot be taken for granted and has to be formally proved (see remarks after Theorem 4.2 in [24] for an appropriate counterexample, cf. also [88]; for the most up to date results on the dependence of $C_{t,f}$ on t and f, see [25]).

Using Egorov property we can introduce one possible definition of the breaking time by asking for the largest $t = t(\hbar)$ for which RHS of (4.8) remains bounded. The corresponding time scale τ_E is usually referred to as the *Ehrenfest time* which had been introduced many years ago [31, 127] when physicists conjectured that for regular systems τ_E should diverge as a function of $\hbar$ at a power-law rate, while for chaotic ones, at a logarithmic rate. It turns out that the conjecture, although intuitively appealing, is extremely hard to verify on rigorous grounds, even for one of the simplest possible models of quantized fully chaotic systems; i.e., toral cat maps. Most of the results confirm the agreement between the quantum and the classical evolutions up to the logarithmic times. There is however still no general result stating that after the logarithmic time is passed, the quantum system diverges in its behavior from its classical counterpart.

Below we briefly review the results obtained in this matter up to the present time. The first interesting observation is that the problem of estimating the Ehrenfest time is closely related to the problem of determining the semiclassical behavior of the eigenstates of the quantum system, on the one hand, and with the problem of finding the quantum period function on the other. The first problem (semiclassical behavior of eigenstates) is usually approached from the following two, slightly different perspectives. In the first approach one hopes to prove so called Quantum Unique Ergodicity (QUE), which states that *all* eigenstates of a chaotic quantum propagator equidistribute (i.e., converge weakly to the Lebesgue measure) in the classical limit.

In the second approach one relaxes a little bit the above requirement and aims at proving the so-called Schnirelman property ([118, 33]), which states that if a quantum system has ergodic classical limit, then *most* of its eigenstates (with given energy level) equidistribute in the classical phase-space when $\hbar \to 0$. QUE results are very difficult to prove. The first example of QUE for a quantum dynamical system with ergodic classical limit has been presented by Marklof and Rudnick in their paper [88], published in 2000. The example is however very specific. The authors consider irrational skew translations of the two-torus. The key point which simplifies the matter considerably here is that the classical limit is not only ergodic but also uniquely ergodic, meaning that there is no other than Lebesgue measure on which quantum eigenstates can possibly concentrate on in the classical limit. Thus the problem of proving QUE in this particular case was essentially reduced to the problem of proving the Egorov property. Moreover irrational translations, although ergodic, are not weakly mixing and hence do not represent a typical example of a chaotic system. In fact QUE has not been proved yet for any quantized fully chaotic system [23, p.4], and to the contrary in many cases, including as we will see below all cat maps, it simply does not hold.

As far as cat maps are concerned the "closest" results to QUE had been proved either under the very strong additional restriction that the classical limit ($\hbar = 1/N \to 0$) is taken only over special sequences of reciprocals of primes N (and under the assumption that generalized Riemann Hypothesis holds) [45], or under weaker conditions on $\hbar$ (density one sequence in $\mathbb{N}$) but with quite restrictive number-theoretical conditions on admissible cat maps [78], cf. also [77]. In both cases restrictions aim at exactly the same point - to ensure the absence or sufficiently slow growth of the degeneracies of the corresponding eigenvalues, which corresponds to long (of order N) periods of the quantum propagator. Rapidly growing degeneracies of the eigenspaces of the propagator caused by short (e.g., of order $\ln N$) quantum periods existing

for some particular values of the Planck constants allowed for construction of the sequences of eigenstates which concentrate in the semiclassical limit on measures with a nontrivial pure-point component. The existence of such sequences - the phenomenon referred to as *strong scarring* - has recently been proved by Faure, Nonnenmacher and De Bièvre in [54] for an *arbitrary* hyperbolic toral automorphism in 2-dim (for some results in higher dimension see [23]). Thus QUE does not hold in these cases. The scarred eigenstates cannot concentrate however totally on pure-point measures. In [53] Faure and Nonnenmacher show that the Lebesgue measure component of the support of any such state must account for at least 1/2 of the total weight (and the bound is sharp). Moreover, the sequences of scarred eigenstates are very exceptional (in the sense that to construct them one needs to choose these exceptional values of $\hbar$ for which the quantum propagator has minimal, i.e., logarithmic in $\hbar$ period). The conclusion is that although QUE in general fails, some version of the Schnirelman property usually holds for general chaotic quantum systems. For cat maps the property had been proved already in [24]. It has been also confirmed in a wide variety of other contexts (e.g. for ergodic geodesic flows on compact Riemannian manifolds [118, 128, 33], for ergodic billiards [58, 130], and recently for maps with mixed dynamics [87]).

As far as the Ehrenfest time is concerned the discovery of the scarred states turned out to be directly connected with the construction of an example in which the breakdown of the classical-quantum correspondence happens sharply on a logarithmic scale. The appropriate example was constructed within the framework of cat maps by Bonechi and De Bièvre in [22]. The main idea was to consider the action of the quantum propagator on coherent states supported on the region in the phase space of the diameter not exceeding $\sqrt{\hbar}$. For times tending to infinity but no faster than $\tau_E = (2\gamma)^{-1}\ln(\hbar^{-1})$ (γ is the Lyapunov exponent of the cat map) the whole support of the state must shrink and in the classical limit the state's evolution concentrates

on the classical orbit of its center - the behavior shared by both classical and quantum evolutions (the Wigner function of the coherent state converges weakly to the delta function). On time scales slightly longer than τ_E strong mixing properties of the classical map prevail and cause the support of the coherent state to stretch and the resulting semiclassical limit corresponds to a constant function over the whole phase-space (the Wigner function of the coherent state converges to 1). For classical dynamics this scenario will continue regardless the length of the time scale. To the contrary, the quantum system is periodic and the initial concentration phenomena must repeat itself once the period of the map is completed. At that point quantum and classical evolutions depart from each other. It is then enough to find a sequence of Planck constants for which the quantum period function is of order $\ln(\hbar)$. This is exactly the minimal possible period and as was mentioned above the appropriate sequence can be constructed for any cat map yielding the desired logarithmic asymptotics of the breaking time.

A few words of caution are necessary here. First of all, the above described phenomena need not reflect general properties of quantum systems including cat maps. Indeed, according to earlier described results by Keating, Kurlberg and Rudnick (see 4.2 and 4.3) for a great majority of the values of $\hbar$, quantum period is much longer than $\ln(\hbar)$ and hence for 'most' of classical limits the phenomenon may not be visible on this time scale at all. As the time progresses more and more trajectories will start to complete their periods and the breaking of the correspondence, understood as a statistical phenomenon, may happen on longer time scales.

It is also important to distinguish here between two different semiclassical approaches to the question of determining when the breakdown between classical and quantum mechanics occurs. The above-described results belong to the category of direct 'quantum-classical' limits.

Another approach is to study the validity of semiclassical approximations to

quantum propagators. Here the situation is different and the correspondence on times scales much longer than logarithmic have been observed in several cases. For numerical results in these directions see [121, 122, 101]. These results cannot shed any light on the question of the breaking time for cat maps, however, since in this particular case the semiclassical approximation to the quantum propagator is exact and hence no breaking time can be observed through this analysis.

A general conclusion is that depending on the observed property and the level of the intensity of the phenomenon on which the correspondence is tested one can expect that a whole spectrum of different breaking times may exist ranging from logarithmic to power-law scales in $\hbar$. In Section 5.4, Corollary 5.18 (point IV) we give an example of the situation when the quantum and classical noisy toral automorphisms behave in the same way from their dissipative properties point of view, while the evolutions have already exceeded the Ehrenfest time. It would be of particular interest now to see whether 'CNT-entropic' breaking time coincides with 'ALF-entropic' breaking time and how these two are related to Ehrenfest and dissipation time scales. The results which we are going to derive in the next chapter are closely related to this question. The complete solution is however still elusive.

Chapter 5

Quantum dissipation time

The main goal of this chapter is to introduce the notion of quantum dissipation time and to study its semiclassical and noisy asymptotics for quantized toral maps. The first step is to construct appropriate quantum model of the phase space (Section 5.1) and of the dynamics (Section 5.2) so that it can be considered as a quantization of the classical systems studied in Part I. In the second step (Section 5.4) we will then focus on definitions and semiclassical analysis of the corresponding quantum notions.

5.1 Quantization on the torus. The kinematics

In this section we present in a systematic and rigorous way kinematic step of the quantization of classical systems on the torus. That is, we construct appropriate Hilbert spaces of pure states and algebras of observables. In the following section we will concentrate on the dynamical step i.e. on quantization of canonical toral maps. The presentation is based on the approach by Bouzouina and De Biévre [24] and generalizes it in two directions:

1. We quantize the systems with arbitrary phase-space dimension (for equivalent approaches to quantization in multidimensional setting see e.g. [108, 61]).

2. The quantization scheme is extended in such a way that both finite and infinite dimensional settings are included as particular cases.

Throughout this chapter we will use the terms Model I and Model II to refer respectively to finite and infinite dimensional quantization schemes.

5.1.1 Weyl quantization on $\mathbb{R}^d$

We briefly recall the standard Weyl quantization of systems with d degrees of freedom and with $\mathbb{R}^{2d}$ as a classical phase space (for systematic presentation see [5]).

As before, h will denote the Planck constant. Whenever convenient we will also use the notation $\hbar = \frac{h}{2\pi}$.

Consider usual quantized position and momentum operators

$$\boldsymbol{Q} = (Q_1, ..., Q_d), \qquad \boldsymbol{P} = (P_1, ..., P_d)$$

on the Hilbert space $L^2(\mathbb{R}^d)$ of square integrable wave functions

$$Q_j\psi(\boldsymbol{x}) = \boldsymbol{x}_j\psi(\boldsymbol{x}), \qquad P_j\psi(\boldsymbol{x}) = -i\hbar\frac{\partial\psi}{\partial\boldsymbol{x}_j}(\boldsymbol{x}).$$

The domains of these operators contain the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and hence are dense in $L^2(\mathbb{R}^d)$. Although all are essentially selfadjoint, they do not possess a common domain of selfadjointness. In Weyl quantization one considers exponentiated versions of the above operators, which helps to avoid the domain problems and, more importantly, introduces in a natural way the notion of quantum translation operators

$$U_{\boldsymbol{q}} = e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\boldsymbol{P}}, \qquad V_{\boldsymbol{p}} = e^{\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{Q}}.$$

The explicit action of $U_{\boldsymbol{q}}$ and $V_{\boldsymbol{p}}$ on $L^2(\mathbb{R}^d)$ is given by

$$(U_{\boldsymbol{q}}\psi)(\boldsymbol{x}) = \psi(\boldsymbol{x}-\boldsymbol{q}), \qquad (V_{\boldsymbol{p}}\psi)(\boldsymbol{x}) = e^{\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{x}}\psi(\boldsymbol{x}).$$

The commutation relations for $U_{\boldsymbol{q}}$ and $V_{\boldsymbol{p}}$ follow easily

$$U_{\boldsymbol{q}}V_{\boldsymbol{p}}\psi(\boldsymbol{x}) = e^{\frac{i}{\hbar}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{q})}\psi(\boldsymbol{x}-\boldsymbol{q}), \qquad V_{\boldsymbol{p}}U_{\boldsymbol{q}}\psi(\boldsymbol{x}) = e^{\frac{i}{\hbar}\boldsymbol{p}\cdot\boldsymbol{x}}\psi(\boldsymbol{x}-\boldsymbol{q}),$$

and hence

$$U_{\boldsymbol{q}} V_{\boldsymbol{p}} = e^{-\frac{i}{\hbar}\boldsymbol{q}\cdot\boldsymbol{p}} V_{\boldsymbol{p}} U_{\boldsymbol{q}}. \tag{5.1}$$

Using BCH formula and the fact that $[\boldsymbol{p}\cdot\boldsymbol{Q}, \boldsymbol{q}\cdot\boldsymbol{P}] = i\hbar\boldsymbol{q}\cdot\boldsymbol{p}\mathbf{1}$ we also get

$$U_{\boldsymbol{q}} V_{\boldsymbol{p}} = e^{-\frac{i}{2\hbar}\boldsymbol{q}\cdot\boldsymbol{p}} e^{\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{Q}-\boldsymbol{q}\cdot\boldsymbol{P})}.$$

Let $\boldsymbol{v} = (\boldsymbol{q}, \boldsymbol{p}) \in \mathbb{R}^{2d}$. The Weyl translation operators $T_{\boldsymbol{v}}$ are defined as the following symmetrized products of $U_{\boldsymbol{q}}$ and $V_{\boldsymbol{p}}$

$$T_{\boldsymbol{v}} = e^{\frac{i}{2\hbar}\boldsymbol{q}\cdot\boldsymbol{p}} U_{\boldsymbol{q}} V_{\boldsymbol{p}} = e^{-\frac{i}{2\hbar}\boldsymbol{q}\cdot\boldsymbol{p}} V_{\boldsymbol{p}} U_{\boldsymbol{q}} = e^{\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{Q}-\boldsymbol{q}\cdot\boldsymbol{P})}.$$

Using the notation convention $\boldsymbol{X} = (\boldsymbol{Q}, \boldsymbol{P})$ and $\boldsymbol{v} \wedge \boldsymbol{X} = \boldsymbol{p}\cdot\boldsymbol{Q} - \boldsymbol{q}\cdot\boldsymbol{P}$ the Weyl operators can be written in a compact form

$$T_{\boldsymbol{v}} = e^{\frac{i}{\hbar}\boldsymbol{v}\wedge\boldsymbol{X}}. \tag{5.2}$$

$T_{\boldsymbol{v}}$ can naturally be extended to the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.
The explicit action of $T_{\boldsymbol{v}}$ on a (generalized) wave function $\psi \in \mathcal{S}'(\mathbb{R}^d)$ is given by

$$(T_{\boldsymbol{v}}\psi)(\boldsymbol{x}) = e^{\frac{i}{2\hbar}\boldsymbol{q}\cdot\boldsymbol{p}} e^{\frac{i}{\hbar}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{q})} \psi(\boldsymbol{x}-\boldsymbol{q}) = e^{\frac{i}{\hbar}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{q}/2)} \psi(\boldsymbol{x}-\boldsymbol{q}) \tag{5.3}$$

and yields the following property

$$T_{\boldsymbol{v}} T_{\boldsymbol{v}'} = e^{\frac{i}{2\hbar}\boldsymbol{v}\wedge\boldsymbol{v}'} T_{\boldsymbol{v}+\boldsymbol{v}'}, \tag{5.4}$$

which in turn implies Weyl-type Canonical Commutation Relations

$$T_{\boldsymbol{v}} T_{\boldsymbol{v}'} = e^{\frac{i}{\hbar}\boldsymbol{v}\wedge\boldsymbol{v}'} T_{\boldsymbol{v}'} T_{\boldsymbol{v}}. \tag{5.5}$$

5.1.2 The space of pure states on $\mathbb{T}^d$.

In this section we determine a quantum analog of the notion of periodic phase-space of a classical system by introducing the space of generalized quasi-periodic wave functions (distributions).

The idea of quantization on the torus will be reflected here in the requirement that the wave function be quasiperiodic in its position or momentum representation.

A distribution $\psi \in \mathcal{S}'(\mathbb{R}^d)$ is called quasiperiodic in position representation if there exists a constant $\boldsymbol{\theta}_p \in \mathbb{T}^d$ such that for any $\boldsymbol{m}_1 \in \mathbb{Z}^d$

$$\psi(\boldsymbol{q}+\boldsymbol{m}_1) \;=\; e^{2\pi i \boldsymbol{\theta}_p \cdot \boldsymbol{m}_1}\psi(\boldsymbol{q}).$$

The set of all such distributions will be denoted by $\mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$.

Similarly a distribution $\psi \in \mathcal{S}'(\mathbb{R}^d)$ is called quasiperiodic in momentum representation if there exists a constant $\boldsymbol{\theta}_q \in \mathbb{T}^d$ such that for any $\boldsymbol{m}_2 \in \mathbb{Z}^d$

$$(\mathcal{F}_h\psi)(\boldsymbol{p}+\boldsymbol{m}_2) \;=\; e^{-2\pi i \boldsymbol{\theta}_q \cdot \boldsymbol{m}_2}(\mathcal{F}_h\psi)(\boldsymbol{p}),$$

where $\mathcal{F}_h$ denotes the quantum Fourier transform

$$(\mathcal{F}_h\psi)(\boldsymbol{p}) = \frac{1}{h^{d/2}}\int_{\mathbb{R}^d}\psi(\boldsymbol{q})e^{-2\pi i \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{h}}d\boldsymbol{q}. \tag{5.6}$$

The corresponding set will be denoted by $\mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q)$.

In our approach the space of all admissible pure states denoted by $\mathcal{H}_h(\boldsymbol{\theta})$ of the quantum system on the torus will always be understood as a linear space generated by (not necessary all) elements of $\mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p) \cup \mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q)$ and equipped, according to usual requirements of quantum mechanics, with some Hilbert structure. The choice of the space will depend on the model one intends to work with.

We note that if $\psi \in \mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$ then its momentum representation $\mathcal{F}_h\psi$ has a discrete uniformly h-spaced and $\boldsymbol{\theta}_p$-shifted support and hence is represented by a Dirac delta comb (or more generally brush) of the form

$$\mathcal{F}_h\psi(\boldsymbol{p}) = h^{d/2}\sum_{\boldsymbol{s}\in\mathbb{Z}^d} c_{\boldsymbol{s}}\delta_{h(\boldsymbol{s}+\boldsymbol{\theta}_p)}(\boldsymbol{q}),$$

where $c = \{c_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Z}^d}$ denotes a sequence of complex numbers. Thus any distribution $\psi \in \mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$ is uniquely determined by a triple $(c, h, \boldsymbol{\theta}_p)$. We will sometimes write $\psi = (c, h, \boldsymbol{\theta}_p)$. Similar remark applies to any element of $\mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q)$.

One easily notices that for any $\psi \in \mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$ and any $\boldsymbol{m}_1 \in \mathbb{Z}^d$

$$T_{(\boldsymbol{m}_1,0)}\psi = e^{-2\pi i \boldsymbol{\theta}_p \cdot \boldsymbol{m}_1}\psi \tag{5.7}$$

Similarly for any $\psi \in \mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q)$ and any $\boldsymbol{m}_2 \in \mathbb{Z}^d$

$$T_{(0,\boldsymbol{m}_2)}\psi = e^{2\pi i \boldsymbol{\theta}_q \cdot \boldsymbol{m}_2}\psi. \tag{5.8}$$

That is, the spaces $\mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$ and $\mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q)$ consists of eigenstates of integral translations.

5.1.3 The algebra of observables on $\mathbb{T}^d$.

In this and subsequent sections we will frequently use the following terminology.

Definition 5.1 *An algebra $\mathcal{A}$ is called*

i) *-algebra if it is equipped with an involution $^* : \mathcal{A} \mapsto \mathcal{A}$, $A^{**} = A$, $A \in \mathcal{A}$.

ii) B^*-algebra if it has a Banach space structure and $\|AB\| \leq \|A\|\|B\|$.

iii) C^*-algebra if it is a B^*-algebra that satisfies $\|A^*A\| = \|A\|^2$, for all $A \in \mathcal{A}$.

iv) H^*-algebra if it is a B^*-algebra with an inner product satisfying

$$\langle A, BC^* \rangle = \langle AC, B \rangle = \langle C, A^*B \rangle$$

(e.g. $\langle A, B \rangle = \tau(A^*B)$ if τ is a faithful tracial state).

v) W^*-algebra (von Neumann algebra) if $\mathcal{A}'' = \mathcal{A}$, where $\mathcal{A}''$ is a bicommutant.

The Weyl quantization on the torus will consist in the restriction of the action of Weyl translations from the whole $\mathcal{S}'(\mathbb{R}^d)$ to $\mathcal{H}_h(\boldsymbol{\theta})$.

This restriction is well defined only if translations $T_{\boldsymbol{v}}$ preserve $\mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$ and $\mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q)$ spaces. In view of (5.7) and (5.8) the condition is equivalent to the commutativity of $T_{\boldsymbol{v}}$ with integer translations. The latter condition can be stated as follows

Proposition 5.2

$T_{\boldsymbol{v}} : \mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p) \mapsto \mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$ iff $\boldsymbol{v} \in \mathbb{R}^d \times h\mathbb{Z}^d$.

$T_{\boldsymbol{v}} : \mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q) \mapsto \mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q)$ iff $\boldsymbol{v} \in h\mathbb{Z}^d \times \mathbb{R}^d$.

Remark 5.3 *If $\mathcal{H}_h(\boldsymbol{\theta})$ has nontrivial intersection with $\mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$ and $\mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q)$ then $T_{\boldsymbol{v}} : \mathcal{H}_h(\boldsymbol{\theta}) \mapsto \mathcal{H}_h(\boldsymbol{\theta})$ iff $\boldsymbol{v} \in h\mathbb{Z}^{2d}$.*

Motivated by the above considerations we define microscopic quantum phase-space translations acting on the space $\mathcal{H}_h(\boldsymbol{\theta})$

$$W_{\boldsymbol{k}} := T_{h\boldsymbol{k}} = e^{2\pi i \boldsymbol{k}\wedge \boldsymbol{X}}. \tag{5.9}$$

The operators $W_{\boldsymbol{k}}$ are indexed by the elements of the integral lattice $\mathbb{Z}^{2d}$ and can be thought of as quantum counterparts of classical Fourier modes. The corresponding commutation relations are now given by

$$W_{\boldsymbol{k}}W_{\boldsymbol{m}} = e^{\pi i h \boldsymbol{k}\wedge\boldsymbol{m}} W_{\boldsymbol{k}+\boldsymbol{m}}, \tag{5.10}$$

$$W_{\boldsymbol{k}}W_{\boldsymbol{m}} = e^{2\pi i h \boldsymbol{k}\wedge\boldsymbol{m}} W_{\boldsymbol{m}}W_{\boldsymbol{k}}. \tag{5.11}$$

Note that the placement of the Planck constant has changed after this rescaling (cf. (5.4) and (5.5)).

The formal algebra of observables of our quantum system is generated by the set of operators $\{W_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^{2d}}$ acting on $\mathcal{H}_h(\boldsymbol{\theta})$ and will be denoted by $\mathcal{A}_h(\boldsymbol{\theta})$. We note that $\mathcal{A}_h(\boldsymbol{\theta})$ is a *-algebra with the involution defined by $W_{\boldsymbol{k}}^* = W_{-\boldsymbol{k}}$. Equipped with the standard operator norm $\mathcal{A}_h(\boldsymbol{\theta})$ would become a C^*-algebra and its weak closure - a W^*-algebra.

The reference state defined on the generators of $\mathcal{A}_h(\boldsymbol{\theta})$ by

$$\tau(W_{\boldsymbol{k}}) = \delta_{\boldsymbol{k},0}$$

corresponds to the standard Lebesgue measure in the classical system and can be uniquely extended to a tracial state on the whole algebra $\mathcal{A}_h(\boldsymbol{\theta})$.

In our approach, instead of studying $\mathcal{A}_h(\boldsymbol{\theta})$ as a C^* or W^*-algebras, we prefer to take advantage of its natural H^*-algebraic structure, associated with the state τ. If τ is faithful the inner product on $\mathcal{A}_h(\boldsymbol{\theta})$ is introduced as follows

$$\langle A, B\rangle = \tau(A^*B).$$

In particular

$$\langle W_{\boldsymbol{k}}, W_{\boldsymbol{m}} \rangle = \delta_{\boldsymbol{k},\boldsymbol{m}}$$

and hence automatically $\{W_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^{2d}}$ becomes an orthonormal basis for $\mathcal{A}_h(\boldsymbol{\theta})$.

This completes the discussion of the kinematic step of our general quantization scheme. Below we derive as particular cases two models described in Chapter 4. We call them here Model I and Model II.

5.1.4 Model I

In this section we derive the original Hannay-Berry model including all its later developments and generalizations described in Chapter 4. To this end we set

$$\mathcal{H}_h^I(\boldsymbol{\theta}) := \mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p) \cap \mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q).$$

Thus a distribution $\psi \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $\mathcal{H}_h^I(\boldsymbol{\theta})$ iff for any $\boldsymbol{m} \in \mathbb{Z}^{2d}$,

$$\begin{aligned} \psi(\boldsymbol{q}+\boldsymbol{m}_1) &= e^{2\pi i \boldsymbol{\theta}_p\cdot\boldsymbol{m}_1}\psi(\boldsymbol{q}), \\ (\mathcal{F}_h\psi)(\boldsymbol{p}+\boldsymbol{m}_2) &= e^{-2\pi i \boldsymbol{\theta}_q\cdot\boldsymbol{m}_2}(\mathcal{F}_h\psi)(\boldsymbol{p}). \end{aligned}$$

The parameters $\boldsymbol{\theta}_p, \boldsymbol{\theta}_q$ are sometimes called Floquet or Bloch angels. It is well known (see e.g. [24]) that the set of all distributions satisfying the above mutual quasiperiodicity conditions contains nonzero elements iff

$$h = \frac{1}{N}, \tag{5.12}$$

where $N \in \mathbb{Z}_+$. In literature (5.12) is sometimes referred to as the Bohr-Sommerfield condition. Thus whenever we discuss the Model I we assume that (5.12) holds.

We pause for a while to introduce some notation. Let

$$J_- := \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \qquad J_+ := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},$$

where I denotes the identity matrix on $\mathbb{R}^d$.

For any pair of vectors $\boldsymbol{v}, \boldsymbol{v}' \in \mathbb{R}^{2d}$ we set

$$\boldsymbol{v} \wedge \boldsymbol{v}' := \boldsymbol{v} J_- \boldsymbol{v}', \qquad \boldsymbol{v} \vee \boldsymbol{v}' := \boldsymbol{v} J_+ \boldsymbol{v}'.$$

Of course $\boldsymbol{v} \wedge \boldsymbol{v}'$ is a standard symplectic product on $\mathbb{R}^{2d}$.

Using the above notation quasiperiodicity conditions can be naturally encoded in terms of the action of translation operators

Proposition 5.4 *Let $\psi \in \mathcal{S}'(\mathbb{R}^d)$, then $\psi \in \mathcal{H}_h^I(\boldsymbol{\theta})$ iff for all $\boldsymbol{m} \in \mathbb{Z}^{2d}$,*

$$T_{\boldsymbol{m}} \psi = e^{2\pi i \left(\frac{N}{4} \boldsymbol{m} \vee \boldsymbol{m} + \boldsymbol{m} \wedge \boldsymbol{\theta} \right)} \psi. \tag{5.13}$$

Proof. Obviously if (5.13) holds then ψ is $\boldsymbol{\theta}$-quasiperiodic (consider $\boldsymbol{m} = (\boldsymbol{m}_1, 0)$ and $\boldsymbol{m} = (0, \boldsymbol{m}_2)$). On the other hand if $\psi \in \mathcal{H}_h^I(\boldsymbol{\theta})$ then

$$\begin{aligned} T_{\boldsymbol{m}} \psi &= T_{(\boldsymbol{m}_1,0)+(0,\boldsymbol{m}_2)} \psi = e^{-\pi i N (\boldsymbol{m}_1,0) \wedge (0,\boldsymbol{m}_2)} T_{(\boldsymbol{m}_1,0)} T_{(0,\boldsymbol{m}_2)} \psi \\ &= e^{2\pi i (N/4) \boldsymbol{m} \vee \boldsymbol{m}} e^{-2\pi i \boldsymbol{\theta}_p \cdot \boldsymbol{m}_1} e^{2\pi i \boldsymbol{\theta}_q \cdot \boldsymbol{m}_2} \psi = e^{2\pi i (N/4 \boldsymbol{m} \vee \boldsymbol{m} + \boldsymbol{m} \wedge \boldsymbol{\theta})} \psi. \qquad \blacksquare \end{aligned}$$

The general form of an element of $\mathcal{H}_h^I(\boldsymbol{\theta})$ can easily be determined. One finds (for detailed derivation see Section A.2 of Appendix A) that $\psi \in \mathcal{H}_h^I(\boldsymbol{\theta})$ is necessary a quasiperiodic Dirac delta comb (brush) of the form

$$\psi(\boldsymbol{q}) = \frac{1}{N^{d/2}} \sum_{\boldsymbol{s} \in \mathbb{Z}^d / N} c_{\boldsymbol{s}} \delta_{\boldsymbol{s} + \boldsymbol{\theta}_q / N}(\boldsymbol{q}), \tag{5.14}$$

where $c_{\boldsymbol{s}}$ is a quasiperiodic sequence of arbitrary numbers supported on $\mathbb{Z}^d/N$ lattice and satisfying $c_{\boldsymbol{s}+\boldsymbol{n}} = e^{2\pi i \boldsymbol{\theta}_p \cdot \boldsymbol{n}} c_{\boldsymbol{s}}$. Thus, although $\mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$ and $\mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p)$ are not Hilbert spaces, $\mathcal{H}_h^I(\boldsymbol{\theta})$ can naturally be identified with the Hilbert space $\mathbb{C}^{N^d}$, by introducing the following L^2-norm

$$\|\psi\|_2^2 := \frac{1}{N^d} \sum_{\boldsymbol{s} \in \mathbb{Q}_N^d} |c_{\boldsymbol{s}}|^2.$$

We note that, the crucial difference between the present model and the one considered in the next section lies in the fact that full quasiperiodicity of the state space $\mathcal{H}_h^I(\boldsymbol{\theta})$ implies also quasiperiodicity of quantum Fourier modes. Indeed we have

Proposition 5.5 *For any* $\boldsymbol{m} \in \mathbb{Z}^{2d}$,

$$W_{\boldsymbol{k}+N\boldsymbol{m}} = e^{2\pi i\alpha(\boldsymbol{k},\boldsymbol{m},\boldsymbol{\theta})} W_{\boldsymbol{k}}, \tag{5.15}$$

where

$$\alpha(\boldsymbol{k},\boldsymbol{m},\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{k}\wedge\boldsymbol{m} + \frac{N}{4}\boldsymbol{m}\vee\boldsymbol{m} + \boldsymbol{m}\wedge\boldsymbol{\theta}.$$

Proof. Using Proposition 5.4 we have for any $\psi \in \mathcal{H}_h^I(\boldsymbol{\theta})$,

$$\begin{aligned} W_{\boldsymbol{k}+N\boldsymbol{m}}\psi &= T_{\boldsymbol{k}/N+\boldsymbol{m}}\psi = e^{\pi i\boldsymbol{k}\wedge\boldsymbol{m}} T_{\boldsymbol{k}/N} T_{\boldsymbol{m}}\psi \\ &= e^{\pi i\boldsymbol{k}\wedge\boldsymbol{m}} e^{2\pi i(N/4\boldsymbol{m}\vee\boldsymbol{m}+\boldsymbol{m}\wedge\boldsymbol{\theta})T_{\boldsymbol{k}/N}}\psi = e^{2\pi i\alpha(\boldsymbol{k},\boldsymbol{m},\boldsymbol{\theta})} W_{\boldsymbol{k}}\psi. \qquad \blacksquare \end{aligned}$$

The algebra of observables of Model I will be denoted by $\mathcal{A}_N^I(\boldsymbol{\theta})$. Due to the quasiperiodicity of the set of its generators $\mathcal{A}_N^I(\boldsymbol{\theta})$ is finite dimensional and as a linear space can be identified with $\mathcal{L}(\mathcal{H}_h^I(\boldsymbol{\theta})) \cong \mathcal{M}_{N^d\times N^d} \cong \mathbb{C}^{N^{2d}}$.
We note that $\mathcal{A}_N^I(\boldsymbol{\theta})$ is still a *-algebra with the involution defined by $W_{\boldsymbol{k}}^* = W_{-\boldsymbol{k}}$. The fact that operation $*$ is consistent with the quasiperiodic structure of the set of generators follows from the property $\alpha(-\boldsymbol{k},-\boldsymbol{m},\boldsymbol{\theta}) = -\alpha(\boldsymbol{k},\boldsymbol{m},\boldsymbol{\theta}) \mod 1$.
The tracial state on $\mathcal{A}_N^I(\boldsymbol{\theta})$ can be defined now explicitly

$$\tau(A) = \frac{1}{N^d}\,\mathrm{Tr}\,A.$$

And as above $\{W_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{Z}_N^{2d}\}$ ($\mathbb{Z}_N^{2d} := \mathbb{Z}^{2d} \mod N$) becomes an orthonormal, but this time finite basis for an H^*-structure on $\mathcal{A}_N^I(\boldsymbol{\theta})$.
The H^*-norm on $\mathcal{A}_N^I(\boldsymbol{\theta})$ will be denoted by $\|\cdot\|_{HS}$. One needs to keep in mind that $\|\cdot\|_{HS}$ does not coincide with the standard operator norm, hence $\mathcal{A}_N^I(\boldsymbol{\theta})$ is not (considered here as) a C^*-algebra.

Now we choose the fundamental domain of periodicity $\mathbb{Z}_N^{2d}$ for our quantum Fourier lattice. The choice centered around the origin seems to be the most natural one (cf. [100]). That is, we assume that if $\boldsymbol{k} = (k_1, ..., k_{2d}) \in \mathbb{Z}_N^{2d}$ then for every $j \in \{1, ..., 2d\}$

$$k_j \in \begin{cases} \{-N/2+1, ..., N/2\}, & \text{for } N \text{ even} \\ \{-(N-1)/2+1, ..., (N-1)/2\}, & \text{for } N \text{ odd.} \end{cases}$$

The set $\{W_{\boldsymbol{k}}, \boldsymbol{k} \in \mathbb{Z}_N^{2d}\}$ forms an orthonormal basis for $\mathcal{A}_N(\boldsymbol{\theta})$.

To any classical observable $f \in C^\infty(\mathbb{T}^{2d})$, or more generally $f \in L^2(\mathbb{T}^{2d})$, with $\sum_{\boldsymbol{k}} |\hat{f}(\boldsymbol{k})| < \infty$ there corresponds an element of $\mathcal{A}_N^I(\boldsymbol{\theta})$ i.e. its Weyl quantization, denoted by $Op_N(f)$, and defined in terms of its Fourier expansion

$$Op_N(f) = \sum_{\boldsymbol{k} \in \mathbb{Z}^{2d}} \hat{f}(\boldsymbol{k}) W_{\boldsymbol{k}} = \sum_{\boldsymbol{k} \in \mathbb{Z}_N^{2d}} \left(\sum_{\boldsymbol{m} \in \mathbb{Z}^{2d}} e^{2\pi i \alpha(\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{\theta})} \hat{f}(\boldsymbol{k} + N\boldsymbol{m}) \right) W_{\boldsymbol{k}}.$$

The map Op_N is not invertible. One can define however the isometry $W^P : \mathcal{A}_N^I(\boldsymbol{\theta}) \mapsto L^2(\mathbb{T}^{2d})$ which associates with each observable $A \in \mathcal{A}_N^I(\boldsymbol{\theta})$ its polynomial Weyl symbol

$$W^P(A) = \sum_{\boldsymbol{k} \in \mathbb{Z}_N^{2d}} a_{\boldsymbol{k}} w_{\boldsymbol{k}},$$

where $w_k := e^{2\pi i \boldsymbol{k} \wedge \boldsymbol{x}}$ denote classical Fourier modes and $a_k = \langle W_k, A \rangle$. The operators Op_N and W^P are inverse of each other when the domain of Op_N and the codomain of W^P are restricted to $\mathcal{I}_N = W^P \circ Op_N(L^2(\mathbb{T}^{2d}))$.

5.1.5 Model II

In their original paper [19] the authors did not specify any particular Hilbert space of pure states for their model and instead performed the quantization starting on the algebraic level.

In order to emphasize a close link between the two models within the framework considered here, we first construct the physical Hilbert space for this model and then

show that the natural restriction of Weyl quantization on $\mathcal{S}'(\mathbb{R}^d)$ to this space yields the algebra of observables considered in [19].

First we define the following two Hilbert spaces

$$\mathcal{H}_h^{(q)}(\boldsymbol{\theta}_p) := \{(c, h, \boldsymbol{\theta}_p) \in \mathcal{S}_h^{(q)}(\boldsymbol{\theta}_p) : c \in l^2(\mathbb{Z}^d)\},$$
$$\mathcal{H}_h^{(p)}(\boldsymbol{\theta}_q) := \{(c, h, \boldsymbol{\theta}_q) \in \mathcal{S}_h^{(p)}(\boldsymbol{\theta}_q) : c \in l^2(\mathbb{Z}^d)\}.$$

The values of the parameters $\boldsymbol{\theta}_p$ and $\boldsymbol{\theta}_q$ do not play any significant role in this model and hence we will be working with the following spaces

$$\mathcal{H}_h^{(q)} := \mathcal{H}_h^{(q)}(0), \qquad \mathcal{H}_h^{(p)} := \mathcal{H}_h^{(p)}(0).$$

As as we proved in the previous section, if $(c, h, 0) \in \mathcal{S}_h^{(q)}(0) \cap \mathcal{S}_h^{(p)}(0)$ then either $c \equiv 0$ or $c \notin l^2(\mathbb{Z}^d)\}$, since any such c must be periodic. Thus $\mathcal{H}_h^{(q)} \cap \mathcal{H}_h^{(q)} = \{0\}$.

We then define

$$\mathcal{H}_h^{II} := \mathcal{H}_h^{(q)} \oplus \mathcal{H}_h^{(p)}.$$

In opposition to $\mathcal{H}_h^{I}$, the space $\mathcal{H}_h^{II}$ is infinite dimensional.

Now similarly as in Model I the quantization consists in the restriction of the standard Weyl quantization on $\mathcal{S}'(\mathbb{R}^d)$ to the space $\mathcal{H}_h^{II}$. The construction of the space $\mathcal{H}_h^{II}$ insure that the set of all admissible quantum translations is discrete and coincides with the family of operators $W_{\boldsymbol{k}\boldsymbol{k}\in\mathbb{Z}^{2d}}$ considered in the previous section (see Remark 5.3).

As it was mentioned above (cf. (5.10)) $W_{\boldsymbol{k}}$ satisfy the relations

$$W_{\boldsymbol{k}} W_{\boldsymbol{m}} \;\; = \;\; e^{\pi i h \boldsymbol{k} \wedge \boldsymbol{m}} W_{\boldsymbol{k}+\boldsymbol{m}}.$$

The Planck constant h plays the role of deformation parameter θ considered in [19] (unrelated, of course, to our $\boldsymbol{\theta}$). Putting $h := 2\theta$ one recovers exactly the relations assumed in [19]. The algebra of observables of model II $\mathcal{A}_h^{II}$ can now be defined as a *-algebra generated by elements $W_{\boldsymbol{k}}$. Taking the weak closure (the bicommutant)

in $\mathcal{B}(\mathcal{H}_h^{II})$ yields a W^*-algebra isomorphic to the algebra of observables introduced in [19].

For all $h \geq 0$, $\mathcal{A}_h^{II}$ is always infinite dimensional and hyperfinite i.e. is generated by an ascending sequence of finite dimensional algebras. However the internal structure of $\mathcal{A}_h^{II}$ depends in an essential way on h is rational or not. Before we discuss the classification of $\mathcal{A}_h^{II}$ we need to introduce some definitions [27].

Definition 5.6

i) A W^*-algebra is called a factor if its center is trivial (the algebra is irreducible).

ii) A factor is of type I if it is isomorphic to $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

iii) If dim$(\mathcal{H}) = N$ then the factor is called to be of type I_N.

iv) Two projections P_1, P_2 in a W^*-algebra are said to be equivalent if there exists an element A in the algebra such that $P_1 = A^*A$ and $P_2 = AA^*$.

v) Projection is said to be finite if it is not equivalent to any proper subprojection of itself.

vi) A factor is called type II_1 if the identity operator $\mathbb{1}$ is finite.

If h is rational then $\mathcal{A}_h^{II}$ factorizes into its nontrivial commutative center and a type I_{N^d} factor (matrix algebra)

$$\mathcal{A}_h^{II} = L^2(T^{2d}) \otimes \mathcal{M}_{N^d \times N^d}$$

where N is the smallest integer such that $hN \in \mathbb{Z}_+$ (see [73] and [61] for transparent proofs). Moreover it can be shown (cf. [24]) that in this case The algebras of both models are related by the following decomposition formula

$$\mathcal{A}_h^{II} = \int_{\mathbb{T}^2}^{\oplus} \mathcal{A}_N^I(\boldsymbol{\theta}),$$

If h is irrational $\mathcal{A}_h^{II}$ is a factor of type $II1$ (cf. [113]).

In both cases $\mathcal{A}_h$ admits a faithful tracial state, which in case of rational h factorizes into standard Lebesgue measure on $\mathbb{T}^{2d}$ and the normalized trace on $\mathcal{M}_{N^d \times N^d}$.

Moreover it can be shown that

We end this section with a few remarks regarding the choice of the Hilbert space for the Model II.

Remark 5.7 *The choice of the physical space for Model II is not unique. One can consider e.g. the space* $\tilde{\mathcal{H}}_h^{II} := \mathcal{H}_h^{(q)} \otimes \mathcal{H}_h^{(p)}$ *and the algebra generated by elements* $\mathcal{W}_{\boldsymbol{k}} = W_{\boldsymbol{k}} \otimes W_{\boldsymbol{k}}$.

Remark 5.8 *In contrast to the case of Model I, for all irrational* h*, the algebra of Model II is of type* $II1$ *is never isomorphic to any full (i.e. type I) algebra* $\mathcal{B}(\mathcal{H}_h)$ *regardless the choice of the Hilbert space* $\mathcal{H}_h$.

5.2 Quantization on the torus. The dynamics

In this section we perform the dynamical step in the process of quantization on the torus. We consider arbitrary canonical toral maps i.e. area and orientation preserving homeomorphisms on $\mathbb{T}^{2d}$.

Let Φ denote such a map. As was already mentioned in Section 3.1.1, Φ can be decomposed into the product of three maps $\Phi = F \circ t_{\boldsymbol{v}} \circ \Phi_1$, where $F \in SL(2d, \mathbb{Z})$ is a symplectomorphism, $t_{\boldsymbol{v}}$ denotes a classical translation by vector $\boldsymbol{v}$ and $\Phi_1(\boldsymbol{x}) = \boldsymbol{x} + p(\boldsymbol{x})$, where p is an arbitrary zero-mean, continuous and periodic function.

We assume that Φ_1 represents a time-1 flow map associated with a periodic Hamiltonian. In 2-dimensional case this assumption is equivalent (cf. [70]) to the above requirement that p be of zero mean (w.r.t. the Lebesgue measure on $\mathbb{T}^{2d}$). This can always be achieved by replacing P with $p' = p - \langle p \rangle$ and adjusting accordingly translational component of Φ.

To quantize Φ one first quantizes F, $t_{\boldsymbol{v}}$ and Φ_1 separately. The quantization of Φ is then defined as a composition of corresponding quantum *-automorphisms

$$\mathcal{U}_\Phi = \mathcal{U}_F \mathcal{U}_{t_{\boldsymbol{v}}} \mathcal{U}_1.$$

The quantization procedure will be prescribed in such a way that the correspondence principle will hold. In the case of Model I this will be expressed in terms of so called Egorov property, which states that for every $f \in C^\infty(\mathbb{T}^{2d})$ there exists $C > 0$ such that

$$\|\mathcal{U}(Op_N(f)) - Op_N(Uf)\| \leq \frac{C}{N}, \tag{5.16}$$

where $Uf = f \circ \Phi$ is a classical Koopman operator of Φ.

5.2.1 Quantization of symplectomorphisms

Here we describe the quantization of toral symplectomorphism i.e. symplectic automorphism on $\mathbb{T}^{2d}$. In classical setting the action of a toral automorphism on the algebra of classical observables was defined by means of the Koopman operator given by $U_F f = f \circ F$, where f was an arbitrary observable, usually assumed to be an element of $L^\infty(\mathbb{T}^{2d})$ or $C^\infty(\mathbb{T}^{2d})$. In our approach however it was more convenient to consider the action of U_F on a slightly bigger space $L^2(\mathbb{T}^{2d})$. The natural Hilbert structure of the later allowed us to determine the dynamics by specifying it on the basis of classical Fourier modes $w_{\boldsymbol{k}}(\boldsymbol{x}) = e^{2\pi i \boldsymbol{k} \wedge \boldsymbol{x}}$

$$(U_F w_{\boldsymbol{k}})(\boldsymbol{x}) = e^{2\pi i \boldsymbol{k} \wedge F \boldsymbol{x}} = e^{2\pi i \boldsymbol{k} J_- F \boldsymbol{x}} = e^{2\pi i J_- F^\dagger J_-^\dagger \boldsymbol{k} \wedge \boldsymbol{x}},$$

where $F^\dagger$ denotes the transposed map. Setting $F' = J_- F^\dagger J_-^\dagger$ we get

$$U_F w_{\boldsymbol{k}} = w_{F'\boldsymbol{k}}.$$

The most natural way to define the quantum counterpart of this dynamics is to consider the formal superoperator version $\mathcal{U}_F$ of the classical Koopman operator

$$\mathcal{U}_F W_{\boldsymbol{k}} = W_{F'\boldsymbol{k}}. \tag{5.17}$$

Such dynamics will be well defined i.e. will define a *-automorphism of $\mathcal{A}_h(\boldsymbol{\theta})$ only if the action of $\mathcal{U}_F$ is consistent with its algebraic structure (Weyl commutation rela-

tions). We note that

$$\begin{aligned}
\mathcal{U}_F(W_{\boldsymbol{k}}W_{\boldsymbol{m}}) &= e^{\pi i h \boldsymbol{k}\wedge\boldsymbol{m}}\mathcal{U}_F(W_{\boldsymbol{k}+\boldsymbol{m}}) = e^{\pi i h \boldsymbol{k}\wedge\boldsymbol{m}}W_{F'\boldsymbol{k}+F'\boldsymbol{m}} \\
&= e^{\pi i h \boldsymbol{k}\wedge\boldsymbol{m}}e^{-\pi i h F'\boldsymbol{k}\wedge F'\boldsymbol{m}}W_{F'\boldsymbol{k}}W_{F'\boldsymbol{m}} \\
&= e^{2\pi i \frac{h}{2}(\boldsymbol{k}\wedge\boldsymbol{m}-F'\boldsymbol{k}\wedge F'\boldsymbol{m})}\mathcal{U}_F(W_{\boldsymbol{k}})\mathcal{U}_F(W_{\boldsymbol{m}}).
\end{aligned}$$

Thus in order for $\mathcal{U}_F$ to be a *-automorphism the following condition has to be satisfied

$$F'\boldsymbol{k}\wedge F'\boldsymbol{m} = \boldsymbol{k}\wedge\boldsymbol{m} \mod \frac{2}{h}. \tag{5.18}$$

Since the condition (5.18) has to be valid for arbitrary small h we have

Proposition 5.9 *If a map $F \in SL_\pm(2d,\mathbb{Z})$ is quantizable then it is symplectic.*

Symplecticity is then necessary for quantization regardless the model i.e. the space $\mathcal{H}_h(\boldsymbol{\theta})$ one choses to work with. We note that for symplectic maps $F' = F^{-1}$.

Depending on the choice of $\mathcal{H}_h(\boldsymbol{\theta})$ the condition may be also sufficient. Indeed, this is exactly the case in Model II. In view of the lack of additional conditions on the generators of the algebra $\mathcal{A}_h^{II}$ there are no quantization restrictions other than symplecticity of the map.

In some cases, however, the structure of $\mathcal{H}_h(\boldsymbol{\theta})$ impose additional relations on the generators of the algebra $\mathcal{A}_h(\boldsymbol{\theta})$ and then additional conditions are needed. This is the case in Model I, where the assumption of qasiperiodicity in both position and momentum resulted in qasiperiodicity of algebra.

Thus in this case we have to ensure the compatibility of the action of $\mathcal{U}_F$ with the quasiperiodic structure of $\mathcal{A}_N^I(\boldsymbol{\theta})$. To this end we note that for a given quantum Fourier mode $W_{\boldsymbol{k}+N\boldsymbol{m}}$, one can compute the value of $\mathcal{U}_F W_{\boldsymbol{k}+N\boldsymbol{m}}$ in the following two, in general different, ways:

- on one hand using linearity of F' we have

$$\mathcal{U}_F W_{\boldsymbol{k}+N\boldsymbol{m}} = W_{F'\boldsymbol{k}+NF'\boldsymbol{m}} = e^{2\pi i\alpha(F'\boldsymbol{k},F'\boldsymbol{m},\boldsymbol{\theta})}W_{F'\boldsymbol{k}}$$

- on the other hand, by linearity of $\mathcal{U}_F$

$$\mathcal{U}_F W_{\boldsymbol{k}+N\boldsymbol{m}} = e^{2\pi i\alpha(\boldsymbol{k},\boldsymbol{m},\boldsymbol{\theta})}\mathcal{U}_F W_{\boldsymbol{k}} = e^{2\pi i\alpha(\boldsymbol{k},\boldsymbol{m},\boldsymbol{\theta})} W_{F'\boldsymbol{k}}.$$

That is, for every $\boldsymbol{k}$ and $\boldsymbol{m}$ in $\mathbb{Z}^{2d}$ we must have

$$\alpha(F'\boldsymbol{k}, F'\boldsymbol{m}, \boldsymbol{\theta}) = \alpha(\boldsymbol{k}, \boldsymbol{m}, \boldsymbol{\theta}) \mod 1. \tag{5.19}$$

The map F will be called quantizable in Model I if for every $N \in \mathbb{Z}_+$ there exists $\boldsymbol{\theta} \in \mathbb{T}^{2d}$ (possibly depending on N) such that (5.19) holds for all $\boldsymbol{k}$ and $\boldsymbol{m}$ in $\mathbb{Z}^{2d}$. Below we summarize the quantization condition in Model I for general toral automorphisms.

Proposition 5.10 *A toral automorphism represented by $F \in SL_\pm(2d, \mathbb{Z})$ is quantizable iff it is symplectic. For any given $h = N^{-1}$, the corresponding $\boldsymbol{\theta}$ has to satisfy the following condition:*

$$\frac{N}{2}\begin{pmatrix} A\cdot B \\ C\cdot D \end{pmatrix} + F\boldsymbol{\theta} = \boldsymbol{\theta} \mod 1, \tag{5.20}$$

where A, B, C, D denote block-matrix elements of F, that is

$$F = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and $A \cdot B$ denotes a contraction of two matrices into a (column) vector, defined as follows

$$(A \cdot B)_i = \sum_j A_{ij} B_{ij}.$$

The existence of solutions of equations of the type (5.20) is easy to establish. We note that if N is even then one can simply choose $\boldsymbol{\theta} = 0$. The same solution can be chosen whenever the vector

$$\begin{pmatrix} A\cdot B \\ C\cdot D \end{pmatrix}$$

has all even components (and this condition reduces to Hannay and Berry's 'checker-board' condition stated in [63] in $d = 1$ case). Otherwise one considers two cases. If $F - I$ is invertible then for any $\boldsymbol{k} \in \mathbb{Z}^{2d}$

$$\boldsymbol{\theta} = (F - I)^{-1} \left(\frac{N}{2} \begin{pmatrix} A \cdot B \\ C \cdot D \end{pmatrix} + \boldsymbol{k} \right).$$

There are exactly $|\det(F - I)|$ distinct solutions. In particular the solution is unique if $F - I \in SL_{\pm}(2d, \mathbb{Z})$ (in $d = 1$ case $\det(F - I) = 2 - \operatorname{Tr} F$, hence uniqueness holds iff $\operatorname{Tr} F = 1$ or 3). If $F - I$ is singular one can decompose the matrix F into an identity and nonsingular block and construct an appropriate $\boldsymbol{\theta}$ by applying the above considerations to each block separately.

We want remark that in view of the defining condition (5.17), the Egorov property (5.16) is automatically satisfied (with no error term).

We end this section with a few comments about the above quantization conditions. The conditions were imposed to ensure that $\mathcal{U}_F$ is a *-automorphism of the algebra $\mathcal{A}_N^I(\boldsymbol{\theta})$.

In Section 5.5 we will prove the following simple

Proposition 5.11 *Any *-automorphism on finite matrix algebra is inner.*

Thus in view of this proposition our conditions ensure the existence of a physical quantum propagator $\hat{U}_F$ implementing the dynamics on the underlying physical space

$$\mathcal{U}_F A = ad(\hat{U}_F)A = \hat{U}_F^* A \hat{U}_F.$$

It can be shown that $\hat{U}_F$ (which is only determined up to the phase factor) coincides with the propagator introduced by Hannay and Berry in their original paper [63]. For further details regarding this construction we refer the reader to [90].

We also note that there exists a geometric interpretation of these quantization conditions. Indeed, consider the evolution of the Wigner transform of a given wave

function under the cat map dynamics. It is Well known that the Wigner function evolves according to a classical map (see Appendix B) and forms a $(2N)^d \times (2N)^d$ periodic delta brush supported on the half-integer lattice (if N is length of the side of the fundamental domain of its periodicity). Symplecticity insures that the evolved delta brush represents once again a Wigner function.

In case of odd N or when the wave function is $\boldsymbol{\theta}$-quasiperiodic ($\boldsymbol{\theta} \neq 0$) which means that the supporting lattice of Wigner function is shifted by $\boldsymbol{\theta}$ on the coordinate plane, one wants to ensure that this supporting lattice remains on the same place throughout the evolution. The latter property is equivalent to condition (5.20), and can be thought of as the conservation of the initial 'quantum boundary conditions' (see [70]).

5.2.2 Quantization of translations

As explained in Section 5.1.1, a translation $t_{\boldsymbol{v}}$ is quantized on $L^2(\mathbb{R}^d)$ through a Weyl operator $T_{\boldsymbol{v}}$. We have noticed that such quantum translations act inside the algebra $\mathcal{A}_N^I(\boldsymbol{\theta})$ only if $\boldsymbol{v} \in h\mathbb{Z}^{2d}$. If this condition is not satisfied then quantization depends on the model. In case of Model I there are several possibilities to quantize such translations [24]. We will choose the prescription given in [88]: we take the vector $\boldsymbol{v}^{(N)} \in N^{-1}\mathbb{Z}^{2d}$ closest to $\boldsymbol{v}$ (in Eudlidean distance), which can be obtained by taking, for each $j = 1, \dots, 2d$, the component $v_j^{(N)} = \frac{[Nv_j]}{N}$, where $[x]$ denotes the closest integer to x. One then quantizes $t_{\boldsymbol{v}}$ on $\mathcal{H}_N(\boldsymbol{\theta})$ through the restriction of $T_{\boldsymbol{v}^{(N)}}$ on that space (this is the same operator as $W_{[N\boldsymbol{v}]}(N, \boldsymbol{\theta})$). The corresponding *-automorphism on $\mathcal{A}_N(\boldsymbol{\theta})$ is given by

$$\mathcal{U}_{t_{\boldsymbol{v}}} = ad(T_{\boldsymbol{v}^{(N)}}).$$

It was proved in [88] that the Egorov property (5.16) holds for this quantization. In case of Model II the situation is simpler. Even though $T_{\boldsymbol{v}}$ itself does not act as

an inner *-automorphism on $\mathcal{A}_\hbar^{II}$ it introduces an external *-automorphism on this algebra

$$\mathcal{U}_{t_{\boldsymbol{v}}} W_{\boldsymbol{k}} := T_{\boldsymbol{v}}^* W_{\boldsymbol{k}} T_{\boldsymbol{v}} = e^{-2\pi i \boldsymbol{v}\wedge\boldsymbol{k}} W_{\boldsymbol{k}} \tag{5.21}$$

and this automorphism can be taken as a quantization of $t_{\boldsymbol{v}}$. Note that definition given by (5.21) cannot be applied in Model I since the action of $\mathcal{U}_{t_{\boldsymbol{v}}}$ is not consistent with quasiperiodic structure of $\mathcal{A}_N^I(\boldsymbol{\theta})$.

5.2.3 Quantization of time-1 flow maps of periodic Hamiltonians

We present here the quantization in case of Model I. Let Φ_1 denote the time-1 flow map associated with a periodic Hamiltonian $H(\boldsymbol{z}, t)$, meaning that $\Phi_t : \mathbb{T}^{2d} \to \mathbb{T}^{2d}$ satisfies the Hamilton equations:

$$\frac{\partial \Phi_t(\boldsymbol{z})}{\partial t} = \nabla^\perp H(\Phi_t(\boldsymbol{z}), t), \qquad \Phi_0 = I.$$

To quantize Φ_1, one applies the Weyl quantization to the Hamiltonian $H(t)$, obtaining a time-dependent Hermitian operator $Op_{N,\boldsymbol{\theta}}(H(t))$. From there, one constructs the time-1 quantum propagator on $\mathcal{H}_N(\boldsymbol{\theta})$ associated with the Schrödinger equation for $Op_{N,\boldsymbol{\theta}}(H(t))$:

$$U_{N,\boldsymbol{\theta}}(\Phi_1) := \mathcal{T}\, e^{-2\pi i N \int_0^1 Op_{N,\boldsymbol{\theta}}(H(t))\, dt}$$

($\mathcal{T}$ represents the time ordering). As above, the corresponding *-automorphism on $\mathcal{A}_N^I(\boldsymbol{\theta})$ is defined by

$$\mathcal{U}_1 A = ad(U_{N,\boldsymbol{\theta}}(\Phi_1))\, A = U_{N,\boldsymbol{\theta}}^*(\Phi_1) A U_{N,\boldsymbol{\theta}}(\Phi_1).$$

For such a propagator, the Egorov property is holds, yet with a nonzero error (see [109], Theorem IV.10 for the case of a time-independent Hamiltonian on $\mathbb{R}^{2d}$).

5.3 Quantum noise

In this section we construct quantum noise operator for our system. We first recall some standard facts regarding quantum noise in general (cf. [5],[32]).

The influence of a noise on a quantum system is described through the interaction between the system and its environment. If the system is in a state given by a density matrix ρ and the state of the environment is ρ_{env} then the state of the open system (universe) is given by their tensor product $\rho \otimes \rho_{env}$. As a principle the quantum evolution of the whole universe is assumed to be unitary. Thus there exists a unitary operator U such that the evolved jointed density is given by $U\rho \otimes \rho_{env} U^*$. The noisy quantum evolution Γ of the small system is recovered by tracing out the environment

$$\Gamma(\rho) = \mathrm{Tr}_{env}(U\rho \otimes \rho_{env} U^*). \tag{5.22}$$

In general, Γ is not unitary. It is however always (I) trace preserving and (II) completely positive. Complete positivity implies positivity which together with trace preserving property ensures that Γ maps densities into densities.

Usually the environment is not specified explicitly and the noisy evolution of the system is described entirely in terms of some abstract operator Γ acting only on the small system. Although positivity and trace preserving property would suffice to insure that such operator preserves densities, it would not insure its representability in the form (5.22). The sufficient condition for such representation to exists is complete positivity (which provides robustness of positivity of Γ w.r.t. tensor products). Any quantum noise operator is thus characterized by and should satisfy properties (I) and (II).

By Kraus theorem condition (I) is equivalent to the fact that Γ admits the following operation-sum representation

$$\Gamma(\rho) = \sum_k G_k \rho G_k^*, \tag{5.23}$$

where G_k, called operation elements, are arbitrary bounded operators. The above representation is not unique and in some cases it is useful to use continuous parameter (integral) representation

$$\Gamma(\rho) = \int_k dk G_k \rho G_k^*, \tag{5.24}$$

The corresponding dynamics on the observables of the system is given by

$$A \mapsto \sum_k G_k^* A G_k \qquad A \mapsto \int_k dk G_k^* A G_k.$$

Condition (II) is then equivalent to the requirement that the family G_k constitutes operational partition of unity

$$\sum_k G_k^* G_k = Id. \tag{5.25}$$

We now proceed with the construction of the quantum analog of the classical noise operator introduced in Section 2.1.2. We start with Model I and follow the quantization method used in [100]. For given noise generating density g we will use the following notation

$$g_\epsilon(\boldsymbol{x}) := \frac{1}{\epsilon^{2d}} g\left(\frac{\boldsymbol{x}}{\epsilon}\right), \qquad \tilde{g}_\epsilon(\boldsymbol{x}) := \sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} g_\epsilon(\boldsymbol{x}+\boldsymbol{n}).$$

Also the following notation regarding Fourier transform will be utilized

$$\begin{aligned} \hat{g}(\boldsymbol{\xi}) &:= \int_{\mathbb{R}^{2d}} f(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{x}, \\ \hat{\tilde{g}}(\boldsymbol{k}) &:= \int_{\mathbb{T}^{2d}} \tilde{g}(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{k}} d\boldsymbol{x}, \\ \tilde{\hat{g}}(\boldsymbol{x}) &:= \sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} \hat{g}(\boldsymbol{x}+\boldsymbol{n}). \end{aligned}$$

It is easy to check that $\hat{\tilde{g}}_\epsilon(\boldsymbol{k}) = \hat{g}_\epsilon(\boldsymbol{k}) = \hat{g}(\epsilon\boldsymbol{k})$ for $\boldsymbol{k} \in \mathbb{Z}^{2d}$ and hence

$$\tilde{g}_\epsilon(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} \hat{g}(\epsilon\boldsymbol{k}) \boldsymbol{e}_{\boldsymbol{k}}(\boldsymbol{x}).$$

Let us note that the classical noise operator G_ϵ on $L^2(\mathbb{T}^{2d})$ can be represented as follows

$$G_\epsilon f = \int_{\mathbb{T}^{2d}} \tilde{g}_\epsilon(\boldsymbol{v})(t_{\boldsymbol{v}} f)(\boldsymbol{x}) d\boldsymbol{v},$$

where, as before, $t_{\boldsymbol{v}}$ denotes the Frobenius-Perron operator of the classical phase space translation. A natural way of quantizing the noise operator is to formally replace $t_{\boldsymbol{v}}$ with quantum phase space translations $T_{\boldsymbol{v}}$. However due to the discrete character of quantum translations one has to discretize the classical operator before quantization

$$G_\epsilon \mapsto \frac{1}{N^{2d}} \sum_{\boldsymbol{k}\in\mathbb{Z}_N^{2d}} \tilde{g}_\epsilon\left(\frac{\boldsymbol{k}}{N}\right) P_{\frac{\boldsymbol{k}}{N}}$$

and then quantize it by introducing the following superoperator

$$\mathcal{G}_{\epsilon,N} := \frac{1}{Z} \sum_{\boldsymbol{k}\in\mathbb{Z}_N^{2d}} \tilde{g}_\epsilon\left(\frac{\boldsymbol{k}}{N}\right) ad(T_{\frac{\boldsymbol{k}}{N}}) = \frac{1}{Z} \sum_{\boldsymbol{k}\in\mathbb{Z}_N^{2d}} \tilde{g}_\epsilon\left(\frac{\boldsymbol{k}}{N}\right) ad(W_{\boldsymbol{k}}),$$

where, as always, $ad(W_{\boldsymbol{k}})A = W_{\boldsymbol{k}}^* A W_{\boldsymbol{k}}$ for any $A \in \mathcal{A}_N^I(\boldsymbol{\theta})$.

The role of the prefactor $\frac{1}{Z}$ is to insure that $\mathcal{G}_{\epsilon,N}$ is trace preserving. One can easily check (see Appendix 5.5) that $Z = N^{2d}\tilde{g}_{\epsilon N}(0)$ and that moreover

Proposition 5.12 *$\mathcal{G}_{\epsilon,N}$ is a completely positive trace preserving map and admits the following spectral representation on $\mathcal{A}_N^I(\boldsymbol{\theta})$*

$$\mathcal{G}_{\epsilon,N} A = \sum_{\boldsymbol{k}\in\mathbb{Z}_N^{2d}} \gamma_{\epsilon N}(\boldsymbol{k}^\perp) a_{\boldsymbol{k}} W_{\boldsymbol{k}}, \tag{5.26}$$

where

$$\gamma_{\epsilon N}(\boldsymbol{\xi}) := \frac{\tilde{\hat{g}}_{\epsilon N}(N^{-1}\boldsymbol{\xi})}{\tilde{\hat{g}}_{\epsilon N}(0)}, \qquad A \ := \sum_{\boldsymbol{k}\in\mathbb{Z}_N^{2d}} a_{\boldsymbol{k}} W_{\boldsymbol{k}}. \tag{5.27}$$

Defining the subalgebra of observables orthogonal to identity

$$\mathcal{A}_N^0(\boldsymbol{\theta}) = \{A \in \mathcal{A}_N^I(\boldsymbol{\theta}) : a_0 = 0\}$$

and introducing the superoperator norm w.r.t. H^*-norm on $\mathcal{A}_N^0(\boldsymbol{\theta})$

$$\|\mathcal{G}_{\epsilon,N}\| := \sup_{\|A\|_{HS}=1} \|\mathcal{G}_{\epsilon,N} A\|_{HS}$$

one immediately gets

$$\|\mathcal{G}_{\epsilon,N}\| = \max_{0\neq \boldsymbol{k}\in\mathbb{Z}_N^{2d}} \gamma_{\epsilon N}(\boldsymbol{k}),$$

which in particular means that the quantum noise operator acts as a strict contraction on $\mathcal{A}_N^0(\boldsymbol{\theta})$.

We end this section with a brief comment regarding the quantization of the noise operator in the case of Model II. In this case due to the lack of quasiperiodicity conditions and the uniform in $\boldsymbol{v}$ representation of the action of quantum translations on $\mathcal{A}_h^{II}$ (see (5.21)) the quantum noise operator acts in an isomorphic fashion to the classical one: i.e. we can introduce a continuous version of Kraus noise operator

$$\mathcal{G}_{\epsilon,h} := \int_{\mathbb{R}^{2d}} g_\epsilon(\boldsymbol{v}) ad(T_{\boldsymbol{v}}) d\boldsymbol{v},$$

which establishes uniform w.r.t. $\hbar \in \mathbb{R}_+$ isometry between classical and quantum noise operators.

5.4 Semiclassical analysis of quantum dissipation time

For any canonical map of the torus Φ, the full noisy quantum dynamics $\mathcal{T}_{\epsilon,N}$ is defined (in case of Model I) on $\mathcal{A}_N^0(\boldsymbol{\theta})$ by the composition $\mathcal{T}_{\epsilon,N} := \mathcal{G}_\epsilon \mathcal{U}_\Phi$. We will also consider coarse-grained family of quantum operators defined as follows

$$\tilde{\mathcal{T}}_{\epsilon,N}^{(n)} := \mathcal{G}_\epsilon \mathcal{U}_\Phi^n \mathcal{G}_\epsilon. \tag{5.28}$$

We can now introduce the notion of quantum dissipation time.

$$\tau_q(\epsilon, N) := \min\{n \in \mathbb{Z}_+ : \|\mathcal{T}_{\epsilon,N}^n\| < e^{-1}\}, \tag{5.29}$$

$$\tilde{\tau}_c(\epsilon, N) := \min\{n \in \mathbb{Z}_+ : \|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)}\| < e^{-1}\} \tag{5.30}$$

Similarly as in classical case the dissipation time provides an intermediate scale between initial stage of the evolution (where conservative dynamics dominates due to the assumption of negligible contribution from the noise term) and final stage when the noise has already driven the system to its final equilibrium (maximally mixed) state. On the dissipation time scale the contributions from these competing terms are, roughly speaking, balanced. In the following sections we analyse the behavior of the dissipation time for fixed quantum system and for its semiclassical limit. To avoid any confusion we will reserve the symbols T_ϵ, $\tilde{T}_\epsilon$, $\tau_c(\epsilon)$, $\tilde{\tau}_c(\epsilon)$ for classical quantities. We note that in case on Model II, due to the above established isometry between quantum and classical propagators (Section 5.2.1) and noise operators (Section 5.3), quantum dissipation times for this model coincide with their classical counterparts (and do not depend on the value of the Planck constant). Thus in this case no further analysis is necessary. On the opposite, as we will see in the following section, in case of Model I, quantum dynamics differs considerably from its classical version and in order to recover similarities it will be necessary to consider appropriate semiclassical limit.

The main goal of the present section is the semiclassical analysis of the dissipation time of noisy evolution of quantum toral maps. Since semiclassical analysis is meaningful only in the case of Model I, form now on all the considerations are restricted to this case. In order to better understand the need of semiclassical analysis in the case of Model I we start with some simple considerations regarding the behavior of the dissipation time for finite quantum systems with fixed N.

5.4.1 Dissipation time in the 'quantum limit'

We denote by $\mathcal{U}_\Phi$ the quantum Koopman operator associated with a canonical map Φ on the torus $\mathbb{T}^{2d}$. Since $\dim \mathcal{A}_N^I(\boldsymbol{\theta}) < \infty$ there exists a unitary matrix (quantum

propagator) U_N implementing $\mathcal{U}_\Phi$ on $\mathcal{H}_N(\boldsymbol{\theta})$ (cf. Proposition 5.11). That is

$$\mathcal{U}_\Phi A = ad(U_N) = U_N^* A U_N, \qquad A \in \mathcal{A}_N^I(\boldsymbol{\theta}).$$

Since U_N is unitary, there exists a basis of its eigenfunctions $\psi_k^{(N)} \in \mathcal{H}_N(\boldsymbol{\theta})$. For each such function, $\mathcal{U}_\Phi|\psi_k^{(N)}\rangle\langle\psi_k^{(N)}| = |\psi_k^{(N)}\rangle\langle\psi_k^{(N)}|$. Moreover $\mathcal{U}_\Phi \mathbf{1} = \mathbf{1}$, and hence we have

Proposition 5.13 *The degeneracy of unity in the spectrum of any quantum Koopman operator on $\mathcal{A}_N^I(\boldsymbol{\theta})$ is at least of order N^d. For any fixed value of N the corresponding quantum system is nonergodic.*

In Section 2.4, we showed that the classical dissipation time behaves in a power-law fashion in ϵ if the Koopman operator has a nontrivial eigenfunction (if it possesses a modicum of regularity). In finite dimensional quantum setting all observables are 'smooth', since they are represented by a finite Fourier series. Thus one should expect that the existence of nontrivial pure point spectrum of the quantum propagator should lead to slow dissipation. The following Proposition formalizes this intuition.

Proposition 5.14 *Assume that the noise generating density g decays sufficiently fast at infinity: $\exists \gamma > 2d$ s.t. $|g(x)| = \mathcal{O}(|x|^{-\gamma})$ as $|x| \to \infty$.*
Then, for any $N > 0$ and any $\boldsymbol{\theta}$, the quantum noise operator on $\mathcal{A}_N(\boldsymbol{\theta})$ satisfies

$$\|(1 - \mathcal{G}_{\epsilon,N})\| \le C\,(\epsilon N)^\gamma. \tag{5.31}$$

This bound is useful in the limit $\epsilon N \to 0$.
As a result, the quantum dissipation time associated with any quantized map $\mathcal{U}_\Phi$ is bounded from below as $\tau_q(\epsilon, N) \ge C(\epsilon N)^{-\gamma}$, where $C > 0$ is independent of the classical map Φ. Besides, in this regime the coarse-grained quantum dynamics does not undergo dissipation, meaning that $\tilde{\tau}_q(\epsilon, N) = \infty$.

Proof. We use the RHS of the explicit expressions for the eigenvalues $\gamma_{\epsilon,N}(\boldsymbol{k})$ of $\mathcal{G}_{\epsilon,N}$. From the decay assumptions on g, we see that in the limit $\epsilon N \to 0$,

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}-0} g\left(\frac{\boldsymbol{n}}{\epsilon N}\right) \leq (\epsilon N)^{\gamma} \sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}-0} \frac{1}{|\boldsymbol{n}|^{\gamma}}.$$

The sum on the RHS converges because $\gamma > 2d$. Therefore, we get $\gamma_{\epsilon,N}(\boldsymbol{k}) = 1 + \mathcal{O}\big((\epsilon N)^{\gamma}\big)$ uniformly w.r.to $\boldsymbol{k} \in \mathbb{Z}_N^{2d}$. Since $\mathcal{G}_{\epsilon,N}$ is Hermitian, this yields the estimate (5.31).

The lower bound for the quantum dissipation time then follows from the same considerations as in the proof of Theorem 2.12. ■

In view of the above Proposition, the only nontrivial information regarding chaoticity of quantum systems and asymptotics of the dissipation time can be retrieved in appropriate semiclassical limit. We thus turn now to semiclassical analysis of quantum maps.

Following the notation introduced in Section 5.1.3 we denote by $\Pi_{\mathcal{I}_N}$ an orthogonal Galerkin-type projection of $L_0^2(\mathbb{T}^{2d})$ onto its subspace $\mathcal{I}_N$. It is easy to see that the map

$$\sigma_N : \mathcal{B}(\mathcal{A}_N^0(\boldsymbol{\theta})) \ni \mathcal{T} \mapsto W^P \mathcal{T} Op_N \Pi_{\mathcal{I}_N} \in \mathcal{B}(L_0^2(\mathbb{T}^{2d}))$$

defines an isometric embedding of a finite dimensional quantum algebra of superoperators $\mathcal{B}(\mathcal{A}_N^0(\boldsymbol{\theta}))$ into infinite dimensional classical one $\mathcal{B}(L_0^2(\mathbb{T}^{2d}))$.

It has been shown in [100] (see Lemma 1 and its proof there) that for any fixed $\epsilon > 0$, the operators $\sigma_N(\mathcal{T}_{\epsilon,N})$, which are isometric to $\mathcal{T}_{\epsilon,N}$, converge uniformly (i.e. in the norm of $\mathcal{B}(L_0^2(\mathbb{T}^{2d}))$) as $N \to \infty$ to the classical operator T_ϵ. This implies in particular that for any fixed $\epsilon > 0$, and $n \in \mathbb{N}$ the sequence $\sigma_N(\mathcal{T}_{\epsilon,N}^n)$ converges to T_ϵ^n in $\mathcal{B}(L_0^2(\mathbb{T}^{2d}))$ as $N \to \infty$.

The above result, valid for arbitrary quantizable canonical toral map and any compact noise operator, implies that in appropriate semiclassical regime $N \to \infty$ one recovers classical behavior of the dissipation time. More precisely we have

Proposition 5.15 *For any quantizable, canonical map F on the torus and any noise generating function g, quantum dissipation time coincides asymptotically in sufficiently fast classical limit $N = N(\epsilon) \to \infty$ as $\epsilon \to 0$ with its classical counterpart.*

Proof. We have $\|\mathcal{T}^n_{\epsilon,N}\| = \|\sigma_N(\mathcal{T}^n_{\epsilon,N})\| \leq \|\sigma_N(\mathcal{T}^n_{\epsilon,N}) - T^n_\epsilon\| + \|T^n_\epsilon\|$. Now for given ϵ we set $n = \tau_c(\epsilon)$ and choose $N = N(\epsilon)$ such that $\|\sigma_N(\mathcal{T}^n_{\epsilon,N}) - T^n_\epsilon\| < \epsilon$. Thus $\|\mathcal{T}^n_{\epsilon,N}\| \leq \epsilon + e^{-1}$. On the other hand

$$\|T^n_\epsilon\| \leq \|\sigma_N(\mathcal{T}^n_{\epsilon,N}) - T^n_\epsilon\| + \|\sigma_N(\mathcal{T}^n_{\epsilon,N})\| = \|\sigma_N(\mathcal{T}^n_{\epsilon,N}) - T^n_\epsilon\| + \|\mathcal{T}^n_{\epsilon,N}\|$$

Thus for $n = \tau_c(\epsilon) - 1$ and we have $e^{-1} \leq \epsilon + \|\mathcal{T}^n_{\epsilon,N}\|$, which together with the previous inequality establishes the desired result. ■

The above statement, despite its generality gives, no information about the actual behavior of τ_q unless the behavior of the classical one is known.

The behavior of classical dissipation time of general nonlinear maps has been analyzed in Part I of this work. In particular we can use here our result regarding the logarithmic asymptotics of the dissipation time of Anosov systems (see Theorem 3.24). Joining this result with the above Proposition we get

Corollary 5.16 *Let F be a volume preserving C^3 Anosov diffeomorphisms on $\mathbb{T}^d$ and let g be a C^1 noise generating function with fast decay at infinity. Then there exist $A_1, A_2 > 0$ and $\tilde{C} > 0$ such that*

I) Quantum dissipation time of the coarse-grained dynamics satisfies in sufficiently fast semiclassical and small noise limit $\epsilon(N)N \to \infty$ the following estimate

$$A_1 \ln(\epsilon^{-1}(N)) - \tilde{C} \leq \tilde{\tau}_*(\epsilon, N) \leq A_2 \ln(\epsilon^{-1}(N)) + \tilde{C},$$

II) If in addition F has $C^{1+\eta}$-regular foliations and $g \in C^2(\mathbb{R}^d)$ is compactly supported, then in the above specified semiclassical limit the dissipation time of the noisy evolution satisfies for some $C > 0$

$$A_1 \ln(\epsilon^{-1}(N)) - C \leq \tau_*(\epsilon, N) \leq A_2 \ln(\epsilon^{-1}) + C$$

One has to note here however that the crucial from the semiclassical point of view question about the regime i.e. the relation between N and ϵ for which the above asymptotics holds remains in this general setting open. In the next section we answer this question in full details in the case of linear Anosov systems.

5.4.2 Classical limit for quantum toral symplectomorphisms

In this section we analyze linear maps projected on the torus (generalized cat maps). In this case all computations can be carried out explicitly.

To focus attention and avoid unnecessary notational and computational complications we restrict the considerations of this subsection to Gaussian noises (the assumption follows the made in classical case, where only α-stable laws were considered). Thus we set $\hat{g}(\boldsymbol{k}) = e^{-|\boldsymbol{k}|^2}$.

It is easy to see that, as in classical case, for any $A \in \mathcal{A}_N^0(\boldsymbol{\theta})$ and any symplectic toral automorphism

$$\mathcal{T}_{\epsilon,N}^n A = \sum_{0 \neq \boldsymbol{k} \in \mathbb{Z}_N^{2d}} a_k \prod_{l=1}^{n} \gamma_{\epsilon N}((F^{-l}\boldsymbol{k})^{\perp}) W_{F^{-n}\boldsymbol{k}}$$

This yields

$$\|\mathcal{T}_{\epsilon,N}^n\| = \max_{0 \neq \boldsymbol{k} \in \mathbb{Z}_N^{2d}} \prod_{l=1}^{n} \gamma_{\epsilon N}(F^{-l}\boldsymbol{k}) = \max_{0 \neq \boldsymbol{k} \in \mathbb{Z}_N^{2d}} \prod_{l=1}^{n} \gamma_{\epsilon N}(F^{l}\boldsymbol{k}), \tag{5.32}$$

In the above computations we have used the fact that the dissipation time does not depend on the direction of time. Similarly in coarse grained case we get

$$\|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)}\| = \max_{0 \neq \boldsymbol{k} \in \mathbb{Z}_N^{2d}} \gamma_{\epsilon N}(\boldsymbol{k}) \gamma_{\epsilon N}(F^n\boldsymbol{k}). \tag{5.33}$$

We are now in a position to state the main theorem of the present section.

Theorem 5.17 *For arbitrary symplectic, ergodic and diagonalizable $F \in SL(2d, \mathbb{Z})$ and Gaussian noise, one has the following estimates*

I)

$$\tau_q(\epsilon, N) \geq \tau_c(\epsilon), \qquad \tau_q(\epsilon, N) \geq \frac{2}{d}\frac{1}{(N\epsilon)^2}, \qquad \tilde{\tau}_q(\epsilon, N) \geq \tilde{\tau}_c(\epsilon),$$

uniformly in N.

II) There exists $M > 0$ *(cf. 5.40) such that*

$$\tau_q(\epsilon, N) \approx \tau_c(\epsilon) \approx \frac{1}{\hat{h}(F)} \ln(\epsilon^{-1}), \qquad \epsilon \to 0\,, \epsilon N > M,$$

Moreover for any β *satisfying*

$$\beta > \frac{\ln \|F\|}{\hat{h}(F)} + 1 \tag{5.34}$$

one has

$$\tilde{\tau}_q(\epsilon, N) \;\approx\; \tilde{\tau}_c(\epsilon) \approx \frac{1}{\hat{h}(F)} \ln(\epsilon^{-1}), \qquad \epsilon \to 0\,, \epsilon^{\beta} N > 1,$$

where $\hat{h}(F)$ *is a constant equal to minimal dimensionally averaged K-S entropy of* F *(cf. [52]).*

III) In the limit $\epsilon N \to 0$*, the quantum coarse-grained dynamics does not undergo dissipation i.e.* $\tau_q(\epsilon, N) = \infty$*. The noisy dynamics undergoes slow (power-law) dissipation.*

As a direct corollary of the above theorem we get the following relation between spatial (small ϵ and small $\hbar$) and time (dissipation and Ehrenfest) scales for classical-quantum correspondence of noisy quantum dynamics in case of toral automorphisms.

Corollary 5.18 (Dissipation vs. Ehrenfest times)

Under the assumptions of Theorem 5.17 the following relations hold

I)

$$\epsilon \hbar^{-1} \to \infty \Rightarrow \tau_q(\epsilon, N) \lesssim \tau_E,$$

II) There exists $M > 0$ *(see 5.40) such that*

$$\epsilon \hbar^{-1} \sim C > M \Rightarrow \tau_q(\epsilon, N) \sim \tau_E,$$

III) There exists $M_0 \geq d^{-1/2}$ such that

$$\epsilon\hbar^{-1} \sim C < M_0, \quad \tilde{\tau}_*(\epsilon, N) = \infty,$$

IV) If $N\epsilon(\ln(\epsilon^{-1}))^{1/2} \ll 1$ and the classical dynamics has logarithmic dissipation time (i.e. F is ergodic) then

$$\tau_q(\epsilon, N) \gg \tau_c(\epsilon) \gg \tau_E.$$

Proof of Corollary. Statements I) and II) follow immediately from statement II) of the above theorem (and its proof). Statement III) is a direct consequence of the estimate

$$\|\tilde{\mathcal{T}}^{(n)}_{\epsilon,N}\| \geq e^{-d(\epsilon N)^2} \geq e^{-1},$$

which holds true (for all n) whenever $\epsilon N < M_0 := d^{-1/2}$.

The last statement follows from the following estimates. First we note that

$$\tau_q(\epsilon, N) \geq \frac{2}{d}\frac{1}{(N\epsilon)^2} \gg \ln(\epsilon^{-1}) \sim \tau_c(\epsilon)$$

But we also have

$$\tau_c(\epsilon) \sim \ln(\epsilon^{-1}) \geq \ln(\epsilon^{-1}) - \frac{1}{2}\ln\ln(\epsilon^{-1}) \gg \ln N \sim \tau_E.$$

This completes the proof of the corollary ■

To prove the theorem we will need the following estimate (for the proof see Appendix 3.51).

Lemma 5.19

For any $\boldsymbol{\xi} \in \mathbb{R}^{2d}$ denote by $\tilde{\boldsymbol{\xi}}$ the unique vector satisfying $\tilde{\boldsymbol{\xi}}_j \in (-1/2, 1/2]$ for all $j = 1, ..., 2d$ and $\tilde{\boldsymbol{\xi}} = \boldsymbol{\xi} \mod 1$. There exists a constant $C > 0$ such that for all $\sigma > 0$ and all $\boldsymbol{\xi} \in \mathbb{R}^{2d}$,

$$\hat{g}_\sigma(\boldsymbol{\xi}) \leq \hat{g}_\sigma(\tilde{\boldsymbol{\xi}}) \leq \gamma_\sigma(N\boldsymbol{\xi}) \leq \frac{\hat{g}_\sigma(\tilde{\boldsymbol{\xi}})}{\tilde{\hat{g}}_\sigma(0)} + Ce^{-\frac{1}{4}\sigma^2} \leq \hat{g}_\sigma(\tilde{\boldsymbol{\xi}}) + Ce^{-\frac{1}{4}\sigma^2} \tag{5.35}$$

Proof of Theorem.

We start with the proofs of statements I) and III). The proof that quantum dissipation time is never shorter than classical one follows from (5.32) (5.33), Lemma 5.19 and the fact that $\hat{g}_\epsilon(\boldsymbol{\xi}) \leq 1$. Indeed, we have

$$\|\mathcal{T}^n_{\epsilon,N}\| = \max_{0\neq \boldsymbol{k}\in\mathbb{Z}^{2d}_N} \prod_{l=1}^{n} \gamma_{\epsilon N}(F^l\boldsymbol{k}) \geq \sup_{0\neq \boldsymbol{k}\in\mathbb{Z}^{2d}} \prod_{l=1}^{n} \hat{g}_\epsilon(F^l\boldsymbol{k}) = \|T^n_\epsilon\|.$$

Similar estimation holds in coarse-grained case. To show that the remaining assertion in I) and that III) hold we first consider coarse-grained version. Denoting $\boldsymbol{\xi}_0 := N^{-1}\boldsymbol{k}$ and $\boldsymbol{\xi}_n := N^{-1}F^n\boldsymbol{k}$, we have for any $\boldsymbol{k} \in \mathbb{Z}^{2d}_N$,

$$\begin{aligned}\|\tilde{\mathcal{T}}^{(n)}_{\epsilon,N}\| &\geq \gamma_{\epsilon N}(\boldsymbol{k})\gamma_{\epsilon N}(F^n\boldsymbol{k}) \geq \hat{g}_{\epsilon N}(\tilde{\boldsymbol{\xi}_0})\hat{g}_{\epsilon N}(\tilde{\boldsymbol{\xi}_n}) \\ &= e^{-(\epsilon N)^2(|\tilde{\boldsymbol{\xi}_0}|^2+|\tilde{\boldsymbol{\xi}_n}|^2)} \geq e^{-d(\epsilon N)^2}.\end{aligned}$$

And obviously $e^{-d(\epsilon N)^2} \to 1$ as $\epsilon N \to 0$.

In noisy case, continuing the notation $\boldsymbol{\xi}_l := N^{-1}F^l\boldsymbol{k}$, we have

$$\|\mathcal{T}^n_{\epsilon,N}\| \geq \prod_{l=1}^{n} \gamma_{\epsilon N}(F^l\boldsymbol{k}) \geq \prod_{l=1}^{n} \hat{g}_{\epsilon N}(\tilde{\boldsymbol{\xi}_l}) = e^{-(\epsilon N)^2\sum_{l=1}^{n}|\tilde{\boldsymbol{\xi}_l}|^2} \geq e^{-(\epsilon N)^2\frac{dn}{2}}.$$

Thus

$$\tau_q(\epsilon, N) \geq \frac{2}{d}\frac{1}{(\epsilon N)^2}.$$

Now we pass to the proof of the statement II).

The lower bounds for both noisy and coarse-grained versions follow from the general estimate established in point I) and results obtained in classical setting.

We turn now to upper bound computations. First we consider coarse-grained version. In view of (5.33) we have to estimate from above the following product

$$\|\tilde{\mathcal{T}}^{(n)}_{\epsilon,N}\| = \max_{0\neq \boldsymbol{k}\in\mathbb{Z}^{2d}_N} \gamma_{\epsilon N}(\boldsymbol{k})\gamma_{\epsilon N}(F^n\boldsymbol{k}). \tag{5.36}$$

Given β satisfying (5.34) we fix $\delta_\beta > 0$. According to the assumption that $\epsilon^\beta N > 1$, for all $0 < \delta < \delta_\beta$, all sufficiently small $\epsilon > 0$ and all sufficiently big N there exists

$n \in \mathbb{N}$ such that

$$\frac{1}{(1-\delta)\hat{h}(F)} \ln(\epsilon^{-1}) < n < \frac{1}{(1-\delta)\hat{h}(F) \ln \|F\|} \ln(N/2). \tag{5.37}$$

Now, given $\boldsymbol{k}_0 \in \mathbb{Z}_N^{2d}$ we consider two cases

a) $\boldsymbol{k}_0$ generates classical orbit i.e. $\boldsymbol{k}_0, F^n\boldsymbol{k}_0 \in \mathbb{Z}_N^{2d}$. Then in view of Theorem 3.10, for any $\delta' < \delta$, for sufficiently small ϵ and any n satisfying (5.37) we have

$$|\boldsymbol{k}_0|^2 + |F^n\boldsymbol{k}_0|^2 \geq \min_{0 \neq \boldsymbol{k} \in \mathbb{Z}^{2d}} (|\boldsymbol{k}|^2 + |F^n\boldsymbol{k}|^2) > e^{2(1-\delta')\hat{h}(F)n} > e^{2(1-\delta)\hat{h}(F)n} \tag{5.38}$$

Thus for any such $\boldsymbol{k}_0$, there exists $l \in \{0, n\}$ such that

$$|F^l\boldsymbol{k}_0| > e^{(1-\delta)\hat{h}(F)n}.$$

Using (5.35) we arrive at the following upper bound

$$\gamma_{\epsilon N}(\boldsymbol{k})\gamma_{\epsilon N}(F^n\boldsymbol{k}_0) < \gamma_{\epsilon N}(F^l\boldsymbol{k}_0) \leq e^{-\epsilon^2 e^{2(1-\delta)\hat{h}(F)n}} + Ce^{-\frac{1}{4}(\epsilon N)^2} < e^{-1},$$

where the last inequality holds for sufficiently big ϵN.

b) $\boldsymbol{k}_0$ generates non-classical orbit i.e. $F^n\boldsymbol{k}_0 \notin \mathbb{Z}_N^{2d}$. In this case, $|\boldsymbol{k}_0| > e^{(1-\delta)\hat{h}(F)n}$. Indeed otherwise, in view of RHS of (5.37) we would have

$$|F^n\boldsymbol{k}_0| \leq \|F^n\| e^{(1-\delta)\hat{h}(F)n} \leq \frac{N}{2},$$

which would imply $F^n\boldsymbol{k} \in \mathbb{Z}_N^{2d}$.

Thus similarly as in previous case we have for sufficiently big ϵN,

$$\gamma_{\epsilon N}(\boldsymbol{k}_0)\gamma_{\epsilon N}(F^n\boldsymbol{k}_0) < \gamma_{\epsilon N}(\boldsymbol{k}_0) < e^{-\epsilon^2 e^{2(1-\delta)\hat{h}(F)n}} + Ce^{-\frac{1}{4}(\epsilon N)^2} < e^{-1}.$$

The above two cases exhaust all possible values of $\boldsymbol{k}_0$. We then conclude that

$$\|\tilde{\mathcal{T}}_{\epsilon,N}^{(n)}\| < e^{-1},$$

which in view of the definition of dissipation time completes the proof in coarse-grained case.

Now we consider fully noisy case i.e. we need to estimate from above the following product

$$\|\mathcal{T}_{\epsilon,N}^{(n)}\| = \max_{0\neq \boldsymbol{k}\in\mathbb{Z}_N^{2d}} \prod_{l=1}^{n} \gamma_{\epsilon N}(F^l \boldsymbol{k}). \tag{5.39}$$

Let C denote the constant of the RHS of (5.35) and define

$$M := \max\left\{4e^{\hat{h}(F)}\|F\|, \sqrt{4\ln\left(\frac{C}{e^{-1}-e^{-e^{\hat{h}(F)}}}\right)}\right\}. \tag{5.40}$$

We fix $0<\delta<1/2$. Using the fact that $\epsilon N > M$ we obtain for every $\epsilon>0$ the existence of $n\in\mathbb{N}$ such that

$$\frac{1}{(1-\delta)\hat{h}(F)}\ln(\epsilon^{-1}) < n < n+1 < \frac{1}{(1-\delta)\hat{h}(F)}\ln\left(\frac{N}{2\|F\|}\right). \tag{5.41}$$

Now, once again, given $\boldsymbol{k}_0\in\mathbb{Z}_N^{2d}$ we consider two cases.

a) $\boldsymbol{k}_0$ generates classical orbit i.e. $F^l\boldsymbol{k}_0\in\mathbb{Z}_N^{2d}$ for $l=1,...,n$. Then in view of Theorem 3.10, for any $0<\delta'<\delta$, all sufficiently small ϵ, and corresponding n satisfying (5.41) we have

$$\sum_{l=1}^{n}|F^l\boldsymbol{k}_0|^2 \geq \min_{0\neq\boldsymbol{k}\in\mathbb{Z}^{2d}}\sum_{l=1}^{n}|F^l\boldsymbol{k}|^2 > e^{2(1-\delta')\hat{h}(F)(n+1)} > e^{2(1-\delta)\hat{h}(F)(n+1)}. \tag{5.42}$$

Thus for any such $\boldsymbol{k}_0$, there exists $l_0\in\{1,...,n\}$ such that

$$|F^{l_0}\boldsymbol{k}_0| > e^{(1-\delta)\hat{h}(F)(n+1)}.$$

b) $\boldsymbol{k}_0$ generates non-classical orbit i.e. there exists $1\leq l\leq n$ such that $F^l\boldsymbol{k}_0\notin\mathbb{Z}_N^{2d}$. Let l_0 be the largest exponent such that $F^{l_0}\boldsymbol{k}_0\in\mathbb{Z}_N^{2d}$ but $F^{l_0+1}\boldsymbol{k}_0\notin\mathbb{Z}_N^{2d}$. Then

$$|F^{l_0}\boldsymbol{k}_0| > e^{(1-\delta)\hat{h}(F)(n+1)}. \tag{5.43}$$

Indeed, otherwise in view of the RHS of (5.41) one would have

$$|F^{l_0+1}\boldsymbol{k}_0| \leq \|F\||F^{l_0}\boldsymbol{k}_0| \leq \|F\|e^{(1-\delta)\hat{h}(F)(n+1)} < \frac{N}{2},$$

which would imply $F^{l_0+1}\boldsymbol{k}_0\in\mathbb{Z}_N^{2d}$.

Thus in both cases, using (5.35) and (5.40), we get

$$\begin{aligned}\prod_{l=1}^{n}\gamma_{\epsilon N}(F^l\boldsymbol{k}) &< \gamma_{\epsilon N}(F^{l_0}\boldsymbol{k}) \le e^{-\epsilon^2 e^{2(1-\delta)\hat{h}(F)(n+1)}} + Ce^{-\frac{1}{4}(\epsilon N)^2}\\ &< e^{-e^{\hat{h}(F)}} + Ce^{-\frac{M^2}{4}} < e^{-1}.\end{aligned}$$

This completes the proof of statement II) and the whole theorem. ■

5.5 Technical proofs

Proof of Proposition 5.10

In the proof we follow the approach presented in [108].

Since we already know that symplecticity is necessary for quantization it is enough to prove that for given $h = N^{-1}$ and $\boldsymbol{\theta}$ the condition (5.20) is satisfied iff $\mathcal{U}_F$ is a *-automorphism of the algebra $\mathcal{A}_N^I(\boldsymbol{\theta})$ which in view of (5.19) and assumed symplecticity is equivalent to

$$\frac{N}{4}F'\boldsymbol{m}\vee F'\boldsymbol{m} + F'\boldsymbol{m}\wedge\boldsymbol{\theta} = \frac{N}{4}\boldsymbol{m}\vee\boldsymbol{m} + \boldsymbol{m}\wedge\boldsymbol{\theta} \mod 1. \tag{5.44}$$

If N is even then for all $\boldsymbol{m}$, $\frac{N}{4}\boldsymbol{m}\vee\boldsymbol{m}\in\mathbb{Z}$ and hence the above condition is trivially satisfied for $\boldsymbol{\theta}=0$. If N is odd we first note that due to the following identities

$$\begin{aligned}F'\boldsymbol{m}\vee F'\boldsymbol{m} &= (J_-F^\dagger J_-^\dagger)\boldsymbol{m}J_+J_-F^\dagger J_-^\dagger\boldsymbol{m}\\ &= \boldsymbol{m}J_-FJ_-J_+J_-F^\dagger J_-\boldsymbol{m} = \boldsymbol{m}J_-FJ_+F^\dagger J_-\boldsymbol{m}\\ F'\boldsymbol{m}\wedge\boldsymbol{\theta} &= \boldsymbol{m}\wedge F\boldsymbol{\theta},\end{aligned}$$

condition (5.44) can be rewritten as

$$\frac{N}{4}\boldsymbol{m}\vee\boldsymbol{m} + \frac{N}{4}\boldsymbol{m}J_-FJ_+F^\dagger J_-\boldsymbol{m} + \boldsymbol{m}\wedge F\boldsymbol{\theta} = \boldsymbol{m}\wedge\boldsymbol{\theta} \mod 1 \tag{5.45}$$

To see that (5.44) implies (5.20) we take $\boldsymbol{m} = \boldsymbol{e}_j$ ($j\in\{1,...,2d\}$), where $\boldsymbol{e}_j$ denote the standard basis vectors, use the fact that $\boldsymbol{e}_j\vee\boldsymbol{e}_j = 0$ and get

$$\frac{N}{4}\boldsymbol{e}_jJ_-FJ_+F^\dagger J_-\boldsymbol{e}_j + \boldsymbol{e}_j\wedge F\boldsymbol{\theta} = \boldsymbol{e}_j\wedge\boldsymbol{\theta} \mod 1. \tag{5.46}$$

Now we note that

$$\begin{aligned}\frac{N}{4}\boldsymbol{e}_j J_- F J_+ F^\dagger J_- \boldsymbol{e}_j &= \frac{N}{4}\boldsymbol{e}_j \begin{bmatrix} -CD^\dagger - DC^\dagger & CB^\dagger + DA^\dagger \\ -AD^\dagger - BC^\dagger & -AB^\dagger + BA^\dagger \end{bmatrix} \boldsymbol{e}_j \\ &= -\frac{N}{2}\begin{pmatrix} C\cdot D \\ A\cdot B \end{pmatrix} \cdot \boldsymbol{e}_j \end{aligned}$$

and

$$-\frac{N}{2}\begin{pmatrix} C\cdot D \\ A\cdot B \end{pmatrix} \cdot \boldsymbol{e}_j = \frac{N}{2}\boldsymbol{e}_j \wedge \begin{pmatrix} A\cdot B \\ C\cdot D \end{pmatrix} \mod 1.$$

Substituting the above expression in (5.46) immediately yields (5.20).

Now we sketch the proof of the opposite implication. First we rearrange the LHS of (5.45)

$$\begin{aligned} &\frac{N}{4}\sum_{i,j=1}^{2d} m_i m_j \boldsymbol{e}_i J_+ \boldsymbol{e}_j + \frac{N}{4}\sum_{i,j=1}^{2d} m_i m_j \boldsymbol{e}_i J_- F J_+ F^\dagger J_- \boldsymbol{e}_j + \sum_{i=1}^{2d} m_i \boldsymbol{e}_i \wedge F\boldsymbol{\theta} \\ =\ & \frac{N}{4}\sum_{i\neq j=1}^{2d} m_i m_j \boldsymbol{e}_i (J_- F J_+ F^\dagger J_- + J_+)\boldsymbol{e}_j \\ +\ & \frac{N}{4}\sum_{i=1}^{2d} m_i^2 \boldsymbol{e}_i J_- F J_+ F^\dagger J_- \boldsymbol{e}_i + \sum_{i=1}^{2d} m_i \boldsymbol{e}_i \wedge F\boldsymbol{\theta} \end{aligned}$$

Now we note that due to the symplecticity of F

$$\frac{N}{4}\sum_{i\neq j=1}^{2d} m_i m_j \boldsymbol{e}_i J_- F J_+ F^\dagger J_- \boldsymbol{e}_j = \frac{N}{4}\sum_{i\neq j=1}^{2d} m_i m_j \boldsymbol{e}_i J_+ \boldsymbol{e}_j \mod 1$$

Thus it is enough to use the following obvious identity

$$\sum_{i=1}^{2d} m_i^2 \boldsymbol{e}_i J_- F J_+ F^\dagger J_- \boldsymbol{e}_i = \sum_{i=1}^{2d} m_i \boldsymbol{e}_i J_- F J_+ F^\dagger J_- \boldsymbol{e}_i \mod 2,$$

to conclude in view of the first part of the proof the equivalence between (5.44) and (5.20). ■.

Proof of Proposition 5.11.

Denote by $\{\boldsymbol{e}_j\}$ $(j = 1, ..., N)$ the standard basis of $\mathbb{R}^N$ and let $E_{jk} := |\boldsymbol{e}_j\rangle\langle\boldsymbol{e}_k|$, $F_{jk} := \Gamma(E_{jk})$. Γ is inner if there exists $U \in \mathcal{M}_N$ such that $F_{jk} = U^*|\boldsymbol{e}_j\rangle\langle\boldsymbol{e}_k|U = |U^*\boldsymbol{e}_j\rangle\langle U^*\boldsymbol{e}_k| = |\boldsymbol{f}_j\rangle\langle\boldsymbol{f}_k|$, where $\boldsymbol{f}_j = U^*\boldsymbol{e}_j$. It is then enough to show that there exists an orthonormal basis $\{\boldsymbol{f}_j\}$ such that $F_{jk} = |\boldsymbol{f}_j\rangle\langle\boldsymbol{f}_k|$. First we note that $F_{jj} = \Gamma(E_{jj}) = \Gamma(E_{jj}^*) = F_{jj}^*$, $F_{jj} = \Gamma(E_{jj}) = \Gamma(E_{jj}^2) = F_{jj}^2$ and $F_{jj}F_{kk} = \Gamma(E_{jj}E_{kk}) = \delta_{jk}F_{jj}$, which implies that F_{jj} form a set of N mutually orthogonal rank-1 projections. Thus there exists an orthonormal basis $\{\boldsymbol{f}_j\}$ such that $F_{jj} = |\boldsymbol{f}_j\rangle\langle\boldsymbol{f}_j|$. Moreover since $F_{jk} = F_{jj}F_{jk}F_{kk}$ one also finds that $F_{jk} = |\boldsymbol{f}_j\rangle\langle\boldsymbol{f}_k|$. ■.

Proof of Proposition 5.12

Since by definition $\mathcal{G}_{\epsilon,N}$ is already given in Kraus form, it is completely positive [32, 5]. Trace preservation follows from the normalization. The value of the normalization constant can be found as follows

$$Z = \sum_{\boldsymbol{k}\in\mathbb{Z}_N^{2d}} \tilde{g}_\epsilon(N^{-1}\boldsymbol{k}) = \sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} g_\epsilon(N^{-1}\boldsymbol{k}) = N^{2d} \sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} g_{\epsilon N}(\boldsymbol{k}) = N^{2d}\tilde{g}_{\epsilon N}(0).$$

Now using periodicity of $ad(W_{\boldsymbol{k}}$ we get

$$\begin{aligned} \mathcal{G}_{\epsilon,N} &= \frac{1}{Z}\sum_{\boldsymbol{k}\in\mathbb{Z}_N^{2d}} \tilde{g}_\epsilon\left(\frac{\boldsymbol{k}}{N}\right) ad(W_{\boldsymbol{k}}) = \frac{1}{N^{2d}\tilde{g}_{\epsilon N}(0)} \sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} g_\epsilon\left(\frac{\boldsymbol{k}}{N}\right) ad(W_{\boldsymbol{k}}) \\ &= \frac{1}{\tilde{g}_{\epsilon N}(0)} \sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} g_{\epsilon N}(\boldsymbol{k}) ad(W_{\boldsymbol{k}}). \end{aligned}$$

$\mathcal{G}_{\epsilon,N}$ can thus be diagonalized in the basis $\{W_{\boldsymbol{k}}\}$

$$\mathcal{G}_{\epsilon,N}W_{\boldsymbol{k}_0} = \frac{1}{\tilde{g}_{\epsilon N}(0)} \sum_{\boldsymbol{k}\in\mathbb{Z}^{2d}} g_{\epsilon N}(\boldsymbol{k}) e^{2\pi i N^{-1}\boldsymbol{k}_0\wedge\boldsymbol{k}} W_{\boldsymbol{k}_0} = \frac{\tilde{\tilde{g}}_{\epsilon N}(N^{-1}\boldsymbol{k}_0^\perp)}{\tilde{\tilde{g}}_{\epsilon N}(0)} W_{\boldsymbol{k}_0},$$

where in the last inequality we used Poisson summation formula and symmetricity of $g_{\epsilon N}(\boldsymbol{k})$.

Hence for any $\boldsymbol{k} \in \mathbb{Z}^{2d}$, $\mathcal{G}_{\epsilon,N}W_{\boldsymbol{k}} = \gamma_{\epsilon N}(\boldsymbol{k}^\perp)W_{\boldsymbol{k}}$, which yields spectral representation of $\mathcal{G}_{\epsilon,N}$. ■

Proof of Lemma 5.19

Let $\boldsymbol{k}_{1/2} = (1/2, 0, ..., 0) \in \mathbb{R}^{2d}$. There exists a constant C depending only on d such that

$$\begin{aligned}\tilde{\hat{g}}_\sigma(\tilde{\boldsymbol{\xi}}) &\leq \hat{g}_\sigma(\tilde{\boldsymbol{\xi}}) + C\tilde{\hat{g}}_\sigma(\boldsymbol{k}_{1/2}) = \hat{g}_\sigma(\tilde{\boldsymbol{\xi}}) + C\sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} e^{-\sigma^2|\boldsymbol{k}_{1/2}+\boldsymbol{n}|^2} \\ &= \hat{g}_\sigma(\tilde{\boldsymbol{\xi}}) + Ce^{-\frac{\sigma^2}{4}}\sum_{n\in\mathbb{Z}} e^{-\sigma^2 n(n+1)} \sum_{\boldsymbol{m}\in\mathbb{Z}^{2d-1}} e^{-\sigma^2|\boldsymbol{m}|^2}.\end{aligned}$$

Now

$$\sum_{n\in\mathbb{Z}} e^{-\sigma^2 n(n+1)} = 2\sum_{n\geq 0} e^{-\sigma^2 n(n+1)} \leq 2\sum_{n\geq 0} e^{-\sigma^2 n^2} = \sum_{n\in\mathbb{Z}} e^{-\sigma^2 n^2} + 1 \leq 2\sum_{n\in\mathbb{Z}} e^{-\sigma^2 n^2}.$$

Thus

$$\tilde{\hat{g}}_\sigma(\tilde{\boldsymbol{\xi}}) \leq \hat{g}_\sigma(\tilde{\boldsymbol{\xi}}) + 2Ce^{-\frac{\sigma^2}{4}}\sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} e^{-\sigma^2|\boldsymbol{n}|^2} = \hat{g}(\sigma\tilde{\boldsymbol{\xi}}) + 2C\tilde{\hat{g}}_\sigma(0)e^{-\frac{\sigma^2}{4}}$$

and finally

$$\gamma_\sigma(N\boldsymbol{\xi}) = \frac{\tilde{\hat{g}}_\sigma(\tilde{\boldsymbol{\xi}})}{\tilde{\hat{g}}_\sigma(0)} \leq \frac{\hat{g}_\sigma(\tilde{\boldsymbol{\xi}})}{\tilde{\hat{g}}_\sigma(0)} + 2Ce^{-\frac{1}{4}\sigma^2} \leq \hat{g}_\sigma(\tilde{\boldsymbol{\xi}}) + 2Ce^{-\frac{1}{4}\sigma^2}.$$

To prove the other estimate we consider a splitting of the lattice $\mathbb{Z}^{2d}$ into three pairwise disjoint sets $\mathbb{Z}^{2d}_-, \{0\}, \mathbb{Z}^{2d}_+$ such that $\boldsymbol{n} \in \mathbb{Z}^{2d}_+$ iff $-\boldsymbol{n} \in \mathbb{Z}^{2d}_-$. With this notation we have

$$\begin{aligned}\tilde{\hat{g}}_\sigma(\boldsymbol{\xi}) &= \tilde{\hat{g}}_\sigma(\tilde{\boldsymbol{\xi}}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} e^{-\sigma^2|\tilde{\boldsymbol{\xi}}+\boldsymbol{n}|^2} = e^{-\sigma^2|\tilde{\boldsymbol{\xi}}|^2}\sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} e^{-\sigma^2(|\tilde{\boldsymbol{\xi}}+\boldsymbol{n}|^2-|\tilde{\boldsymbol{\xi}}|^2)} \\ &= e^{-\sigma^2|\tilde{\boldsymbol{\xi}}|^2}\sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} e^{-\sigma^2|\boldsymbol{n}|^2}e^{-2\sigma^2\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{n}} \\ &= e^{-\sigma^2|\tilde{\boldsymbol{\xi}}|^2}\left(1 + \sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}_+} e^{-\sigma^2|\boldsymbol{n}|^2}\left(e^{-2\sigma^2\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{n}} + e^{2\sigma^2\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{n}}\right)\right).\end{aligned}$$

Now using the fact that $e^{-2\sigma^2\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{n}} + e^{2\sigma^2\tilde{\boldsymbol{\xi}}\cdot\boldsymbol{n}} \geq 2$ we get

$$\tilde{\hat{g}}_\sigma(\boldsymbol{\xi}) \geq e^{-\sigma^2|\tilde{\boldsymbol{\xi}}|^2}\sum_{\boldsymbol{n}\in\mathbb{Z}^{2d}} e^{-\sigma^2|\boldsymbol{n}|^2} = \hat{g}_\sigma(\tilde{\boldsymbol{\xi}})\tilde{\hat{g}}_\sigma(0).$$

Thus in particular $\gamma_{\epsilon N}(N\boldsymbol{\xi}) \geq \hat{g}_{\epsilon N}(\tilde{\boldsymbol{\xi}})$. ■

Appendix A

The dynamics of cat maps

For completeness we briefly recall in this appendix the most important facts regarding classical cat map dynamics on the plane and its canonical quantization. For simplicity we present the material in 2-dimensional setting.

A.1 Classical dynamics of cat maps

In this section we describe both discrete and continuous-time classical cat map dynamics on the standard phase-plane $\mathbb{R}^2$. In continuous setting we derive the Hamiltonian, the Lagrangian, the Euler-Lagrange equations and the action of the cat map dynamics. This will constitute the basis for canonical quantization presented in the next section.

Discrete dynamics

The classical mechanics of a cat map with 1 degree of freedom is defined on 2 dimensional Euclidean phase space $\mathbb{R}^2$. Any point in the phase space is denoted by (q, p) and represents respectively position and momentum coordinates. The initial state of the system is denoted by (q_0, p_0). The discrete time dynamics is generated by any

ergodic toral automorphism $A \in SL(2, \mathbb{Z})$

$$\begin{bmatrix} q_1 \\ p_1 \end{bmatrix} = A \begin{bmatrix} q_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} q_0 \\ p_0 \end{bmatrix}.$$

In the case of 1 degree of freedom ($d = 1$), considered here, the ergodicity of A is equivalent to its hyperbolicity and is determined by the condition $|\operatorname{Tr} A| > 2$. For simplicity we will even assume that A is positive definite i.e. $\operatorname{Tr} A > 2$.

Continuous dynamics

Continuous version of the dynamics is defined by suspension over the discrete time sequence

$$\begin{bmatrix} q(t) \\ p(t) \end{bmatrix} = A^t \begin{bmatrix} q_0 \\ p_0 \end{bmatrix}.$$

In order to provide an explicit formula for this dynamics we need to compute $A^t = e^{t \ln A}$. We will use the following lemma

Lemma A.1 *Let $A \in \mathcal{M}_n$ and $m_A(z) = \sum_{i=1}^d (z - \lambda_i)^{m_i}$ be a polynomial which satisfies $m_A(A) = 0$ (m_A can be e.g. the minimal or the characteristic polynomial of A). There exist matrices $M_{i,j} \in \mathcal{M}_n$ such that for any sufficiently differentiable function f*

$$f(A) = \sum_{i=1}^{d} \sum_{j=o}^{m_d - 1} f^{(j)}(\lambda_i) M_{i,j}.$$

We will apply the lemma twice. First we apply it to the cat map A and the function ln. We denote by λ the largest eigenvalue of A and take $m_A := (x - \lambda)(x - \lambda^{-1})$. Thus for any continuous function f we have $f(A) = f(\lambda)M_+ + f(\lambda^{-1})M_-$. To find $M_\pm$ we take $f_\pm = x - \lambda^{\pm 1}$ and get

$$A - \lambda^{\pm 1}\mathbb{1} = (\lambda^{\mp 1} - \lambda^{\pm 1})M_{\mp 1} \Rightarrow M_\pm = \frac{1}{\lambda^{\pm 1} - \lambda^{\mp 1}}(A - \lambda^{\mp 1}\mathbb{1})$$

hence

$$f(A) = \frac{f(\lambda) - f(\lambda^{-1})}{\lambda - \lambda^{-1}} A + \frac{f(\lambda^{-1})\lambda - f(\lambda)\lambda^{-1}}{\lambda - \lambda^{-1}} \mathbb{1}.$$

In particular

$$\ln(A) = \frac{\ln\lambda}{\lambda - \lambda^{-1}}(2A - \operatorname{Tr} A\mathbb{1}) = \frac{\ln\lambda}{\lambda - \lambda^{-1}} \begin{bmatrix} a-d & 2b \\ 2c & d-a \end{bmatrix}. \tag{A.1}$$

The prefactor can be written in many different ways, e.g.,

$$\frac{\ln\lambda}{\lambda - \lambda^{-1}} = \frac{h(A)}{\lambda - \lambda^{-1}} = \frac{h(A)}{\sqrt{(\operatorname{Tr} A)^2 - 4}} = \frac{\sinh^{-1}(\frac{1}{2}\sqrt{(\operatorname{Tr} A)^2 - 4})}{\sqrt{(\operatorname{Tr} A)^2 - 4}},$$

where $h(A)$ denotes the KS entropy of A.

Now we apply the lemma to the matrix $\ln A$, denoting by $\mu_1 = \ln\lambda = h(A)$ and $\mu_2 = -\ln\lambda = -h(A)$ its eigenvalues. We have $m_{\ln A} = (x - h(A))(x + h(A))$ and for any f and $f(\ln A) = f(h(A))M_+ + f(-h(A))M_-$. To find $M_\pm$ in this case we take first $f_\pm = x \pm h(A)$

$$\ln A \pm h(A) = \pm 2h(A)M_\pm \Rightarrow M_\pm = \frac{\pm 1}{2h(A)} \ln A + \frac{1}{2}\mathbb{1},$$

hence

$$f(\ln A) = \frac{1}{h(A)} \frac{f(h(A)) - f(-h(A))}{2} \ln A + \frac{f(h(A)) + f(-h(A))}{2} \mathbb{1}.$$

In particular for any even function f, $f(\ln A) = f(h(A))\mathbb{1}$ (e.g. $(\ln A)^2 = h^2(A)\mathbb{1}$) and for any odd function f, $f(\ln A) = \frac{f(h(A))}{h(A)} \ln A$. Taking $f(x) = e^x$ we finally get

$$A^t = \frac{1}{h(A)} \sinh(h(A)t) \ln A + \cosh(h(A)t)\mathbb{1}. \tag{A.2}$$

From the above formula we see that A^t can be considered as a solution of a Cauchy problem for the following matrix-valued ODE

$$\ddot{X} - h^2(A)X = 0,$$

$$X(0) = \mathbb{1}, \quad \dot{X}(0) = \ln A.$$

The fact that A^t indeed satisfies this equation can also be verified directly using the identity $(\ln A)^2 = h^2(A)\mathbb{1}$ and the obvious fact that $\ddot{A}^t = (\ln A)^2 A^t$. This equation will appear later as the Euler-Lagrange equation for this dynamics.

Combining (A.2) with (A.1) we arrive at

$$A^t = \frac{\sinh(h(A)t)}{\sqrt{(\operatorname{Tr} A)^2 - 4}} \begin{bmatrix} a-d & 2b \\ 2c & d-a \end{bmatrix} + \cosh(h(A)t)\mathbb{1}.$$

Now we write down the explicit equations of continuous version of cat map dynamics

$$\begin{aligned} q(t) &= \left(\frac{\sinh(h(A)t)}{\sqrt{(\operatorname{Tr} A)^2 - 4}}(a-d) + \cosh(h(A)t) \right) q(0) + 2b\frac{\sinh(h(A)t)}{\sqrt{(\operatorname{Tr} A)^2 - 4}} p(0) \\ p(t) &= 2c\frac{\sinh(h(A)t)}{\sqrt{(\operatorname{Tr} A)^2 - 4}} q(0) + \left(\frac{\sinh(h(A)t)}{\sqrt{(\operatorname{Tr} A)^2 - 4}}(d-a) + \cosh(h(A)t) \right) p(0). \end{aligned}$$

Hamiltonian and Lagrangian

To find the Hamiltonian of the system we compute

$$\begin{bmatrix} \dot{q}(t) \\ \dot{p}(t) \end{bmatrix} = \ln A A^t \begin{bmatrix} q_0 \\ p_0 \end{bmatrix} = \ln A \begin{bmatrix} q(t) \\ p(t) \end{bmatrix}.$$

In horizontal vector notation this can be rewritten as $(\dot{q}, \dot{p}) = \ln A(q, p)$.

The Hamiltonian $H(p, q)$ satisfies $(\dot{q}, \dot{p}) = \nabla^{\perp} H(q, p) = (H_p(q, p), -H_q(q, p))$. Thus $\nabla^{\perp} H = \ln A$ and

$$H = \frac{h(A)}{\sqrt{(\operatorname{Tr} A)^2 - 4}} \begin{bmatrix} -c & \frac{1}{2}(a-d) \\ \frac{1}{2}(a-d) & b \end{bmatrix},$$

which gives

$$H(q, p) = \frac{h(A)}{\sqrt{(\operatorname{Tr} A)^2 - 4}}(bp^2 - cq^2 + (a-d)pq). \tag{A.3}$$

We note that the Hamiltonian is time independent.

The explicit form of Hamilton equations reads

$$\begin{aligned}\dot{q}(t) &= \frac{\partial H}{\partial p} = \frac{h(A)}{\sqrt{(\operatorname{Tr} A)^2 - 4}}((a-d)q(t) + 2bp(t)) \\ \dot{p}(t) &= -\frac{\partial H}{\partial q} = \frac{h(A)}{\sqrt{(\operatorname{Tr} A)^2 - 4}}(2cq(t) + (d-a)p(t)).\end{aligned}$$

We can now find the Lagrangian using the Legendre transform

$$L(q, \dot{q}) = p\dot{q} - H(q, p) \tag{A.4}$$

We note that since H is time independent, L will not depend explicitly on time either and we drop the time variable. The use of the first Hamilton equation, yields

$$p = \frac{1}{2b}\left(\frac{\sqrt{(\operatorname{Tr} A)^2 - 4}}{h(A)}\dot{q} - (a-d)q\right).$$

Now inserting this formula for p into (A.4) and performing necessary transformations and rearrangements one arrives at

$$L(q, \dot{q}) = \frac{1}{4b}\left(\sqrt{(\operatorname{Tr} A)^2 - 4}\,h(A)q^2 - 2(a-d)\dot{q}q + \sqrt{(\operatorname{Tr} A)^2 - 4}\,(h(A))^{-1}\dot{q}^2\right).$$

Euler-Lagrange equation

Now we find the Euler-Lagrange equation for cat map dynamics. The general form of the Euler-Lagrange equations can be written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

Using the explicit formula for L one gets

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} &= \frac{1}{4b}\left(\sqrt{(\operatorname{Tr} A)^2 - 4}\,(h(A))^{-1}2\ddot{q} - 2(a-d)q\right), \\ \frac{\partial L}{\partial \dot{q}} &= \frac{1}{4b}\left(\sqrt{(\operatorname{Tr} A)^2 - 4}\,(h(A))2q - 2(a-d)\dot{q}\right).\end{aligned}$$

which immediately yields the following Euler-Lagrange equation (Cauchy problem) for our system

$$\ddot{q} - h^2(A)q = 0,$$

$$q(0) = q_0, \quad \dot{q}(0) = v_0,$$

where v_0 denotes the initial velocity.

The action

Now we want to compute the action along the classical path of our dynamics. In terms of Lagrangian the action is defined as

$$S(q_1, q_0) = S(q(t_1), q(t_0)) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t))dt. \tag{A.5}$$

Here we derive only equations for action and compute it for particular choice of time interval, i.e., $t_0 = 0$ and $t_1 = 1$, that is, we find the action associated with the discrete dynamics. To derive action equations we use once again the Legendre transform (A.4) in conjunction with Euler-Lagrange equations to get

$$\frac{\partial L}{\partial \dot{q}} = p, \quad \frac{\partial L}{\partial q} = \dot{p}.$$

Now from (A.5) we get

$$\begin{aligned}\frac{\partial S}{\partial q_1}\dot{q}(t_1) &= \frac{d}{dt_1}S(q(t_1), q(t_0)) = L(q(t_1), \dot{q}(t_1)),\\ \frac{\partial S}{\partial q_0}\dot{q}(t_0) &= \frac{d}{dt_0}S(q(t_1), q(t_0)) = -L(q(t_0), \dot{q}(t_0)).\end{aligned}$$

Dropping the explicit dependence on time and taking derivative w.r.t. $\dot{q}$ we obtain

$$\frac{\partial S}{\partial q_1} = \frac{\partial L(q_1, \dot{q})}{\partial \dot{q}} = p_1, \quad \frac{\partial S}{\partial q_0} = -\frac{\partial L(q_0, \dot{q})}{\partial \dot{q}} = -p_0.$$

Hence the action equations are

$$p_1 = \frac{\partial S}{\partial q_1}, \quad p_0 = -\frac{\partial S}{\partial q_0}.$$

Using these equations one easily checks that for our discrete dynamics

$$S(q_1, q_0) = \frac{1}{2b}(aq_0^2 - 2q_0q_1 + dq_1^2). \tag{A.6}$$

Indeed

$$\begin{aligned}
\frac{\partial S(q_1,q_0)}{\partial q_1} &= \frac{1}{b}(dq_1 - q_0) = \frac{1}{b}(d(aq_0+bp_0)-q_0) \\
&= \frac{(da-1)}{b}q_0 + dp_0 = cq_0 + dp_0 = p_1, \\
\frac{\partial S(q_1,q_0)}{\partial q_0} &= \frac{1}{b}(aq_0 - q_1) = \frac{1}{b}(aq_0 - aq_0 - bp_0) = -p_0.
\end{aligned}$$

A.2 Canonical quantization of cat maps

Notation

Throughout the rest of the appendix the following notation will be used

$$\begin{aligned}
\mathbb{Z}_N &= \mathbb{Z} \mod N = \{0,1,...,N-1\}, \\
\mathbb{Q}_N &= (\mathbb{Z}/N) \mod 1 = \left\{0, \frac{1}{N}, ..., \frac{N-1}{N}\right\}.
\end{aligned}$$

Continuous quantum Fourier transform will be denoted by $\mathcal{F}_h$, i.e.

$$\mathcal{F}_h(\psi)(\boldsymbol{p}) = \frac{1}{h^{d/2}}\int_{\mathbb{R}^d}\psi(\boldsymbol{q})e^{-2\pi i\frac{\boldsymbol{q}\cdot\boldsymbol{p}}{h}}d\boldsymbol{q}. \tag{A.7}$$

With this normalization, Dirac delta function (on $\mathbb{R}^d$) and periodic Dirac delta comb (on $\mathbb{T}^d$) can be written as

$$\begin{aligned}
\delta_{\boldsymbol{a}}(\boldsymbol{q}) &= \frac{1}{h^d}\int_{\mathbb{R}^d} e^{\frac{2\pi i}{h}(\boldsymbol{q}-\boldsymbol{a})\boldsymbol{p}}d\boldsymbol{p} = \int_{\mathbb{R}^d} e^{2\pi i(\boldsymbol{q}-\boldsymbol{a})\boldsymbol{p}}d\boldsymbol{p}, \\
\tilde{\delta}_{\boldsymbol{a}}(\boldsymbol{q}) &:= \sum_{\boldsymbol{n}\in\mathbb{Z}^d}\delta_{\boldsymbol{a}+\boldsymbol{n}}(\boldsymbol{q}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} e^{2\pi i\boldsymbol{n}\cdot(\boldsymbol{q}-\boldsymbol{a})}.
\end{aligned}$$

For any periodic function ψ with period 1 and any vector $\boldsymbol{k}\in h\mathbb{Z}^d$ with $h=1/N$ and $N\in\mathbb{N}$ we denote by $\hat{\psi}(\boldsymbol{k})$ its quantum Fourier coefficients defined as

$$\hat{\psi}(\boldsymbol{k}) = \frac{1}{h^{d/2}}\int_{\mathbb{T}^d}\psi(\boldsymbol{q})e^{-2\pi i\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{h}}d\boldsymbol{q}. \tag{A.8}$$

Then the inverse transform (i.e. the Fourier representation of ψ) is given by

$$\psi(\boldsymbol{q}) = h^{d/2}\sum_{\boldsymbol{k}\in h\mathbb{Z}^d}\hat{\psi}(\boldsymbol{k})e^{2\pi i\frac{\boldsymbol{k}\cdot\boldsymbol{q}}{h}}. \tag{A.9}$$

The normalization in (A.8) implies that the following relation between continuous and periodic Fourier Transforms holds

$$\mathcal{F}_h(\psi)(\boldsymbol{p}) = h^d \sum_{\boldsymbol{k}\in h\mathbb{Z}^d} \hat{\psi}(\boldsymbol{k})\delta_{\boldsymbol{k}}(\boldsymbol{p}) \tag{A.10}$$

Parseval identity in this setting reads

$$\langle \phi, \psi\rangle_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} \overline{\phi}(\boldsymbol{q})\psi(\boldsymbol{q})d\boldsymbol{q} = h^d \sum_{\boldsymbol{k}\in h\mathbb{Z}^d} \overline{\hat{\phi}}(\boldsymbol{k})\hat{\psi}(\boldsymbol{k}) = \langle \hat{\phi}, \hat{\psi}\rangle_{L^2(h\mathbb{Z}^d)}.$$

For a sequence $c = \{c_{\boldsymbol{s}}\}_{\boldsymbol{s}\in\mathbb{Q}_N^d}$ we define its discrete Fourier transform $\hat{c} = \{\hat{c}_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Q}_N^d}$ as

$$\hat{c}_{\boldsymbol{k}} = \frac{1}{N^{d/2}} \sum_{\boldsymbol{s}\in\mathbb{Q}_N^d} c_{\boldsymbol{s}} e^{-2\pi i N \boldsymbol{s}\cdot\boldsymbol{k}}. \tag{A.11}$$

Then

$$c_{\boldsymbol{s}} = \frac{1}{N^{d/2}} \sum_{\boldsymbol{k}\in\mathbb{Q}_N^d} \hat{c}_k e^{2\pi i N \boldsymbol{k}\cdot\boldsymbol{s}}. \tag{A.12}$$

In discrete case the Parseval identity takes the standard form

$$\langle c, d\rangle = \sum_{\boldsymbol{s}\in\mathbb{Q}_N^d} \overline{c}_{\boldsymbol{s}} d_{\boldsymbol{s}} = \sum_{\boldsymbol{k}\in\mathbb{Q}_N^d} \overline{\hat{c}}_{\boldsymbol{k}} \hat{d}_{\boldsymbol{k}} = \langle \hat{c}, \hat{d}\rangle.$$

The relation between continuous and discrete Fourier transforms for periodic delta combs and $h = 1/N$ is given by

$$\mathcal{F}_h\left(\sum_{\boldsymbol{s}\in\mathbb{Z}^d/N} c_{\boldsymbol{s}}\delta_{\boldsymbol{s}}\right) = \sum_{\boldsymbol{k}\in\mathbb{Z}^d/N} \hat{c}_{\boldsymbol{k}}\delta_{\boldsymbol{k}}.$$

Finally we note that

$$\sum_{\boldsymbol{s}\in\mathbb{Z}^d/N} c_{\boldsymbol{s}}\delta_{\boldsymbol{s}} = N^{d/2} \sum_{\boldsymbol{k}\in\mathbb{Z}^d/N} \hat{c}_{\boldsymbol{k}} e^{2\pi i N k\cdot q} \tag{A.13}$$

Quasi-periodic wave functions

Now we are in a position to determine the space of all quasiperiodic wave functions. Let $\boldsymbol{\theta} = (\boldsymbol{\theta}_q, \boldsymbol{\theta}_p) \in \mathbb{T}^{2d}$. The wave function ψ is $\boldsymbol{\theta}$-quasiperiodic (cf. [63, 24]) if for all

$\boldsymbol{m} = (\boldsymbol{m}_1, \boldsymbol{m}_2) \in \mathbb{Z}^{2d}$

$$\psi(\boldsymbol{q} + \boldsymbol{m}_1) = e^{2\pi i \boldsymbol{\theta}_p \cdot \boldsymbol{m}_1} \psi(\boldsymbol{q}), \qquad (\mathcal{F}_h \psi)(\boldsymbol{p} + \boldsymbol{m}_2) = e^{-2\pi i \boldsymbol{\theta}_q \cdot \boldsymbol{m}_2} (\mathcal{F}_h \psi)(\boldsymbol{p}).$$

The second condition gives

$$\frac{1}{h^{d/2}} \int_{\mathbb{R}^d} e^{2\pi i (\boldsymbol{\theta}_q - \boldsymbol{q}/h)\cdot \boldsymbol{m}_2} \psi(\boldsymbol{q}) e^{-2\pi i \frac{\boldsymbol{q}\cdot\boldsymbol{p}}{h}} d\boldsymbol{q} = (\mathcal{F}_h \psi)(\boldsymbol{p}).$$

Using the bijective property of the Fourier transform on $\mathcal{S}'(\mathbb{R}^d)$ we get

$$e^{2\pi i (\boldsymbol{\theta}_q - \boldsymbol{q}/h)\cdot \boldsymbol{m}_2} \equiv 1,$$

which gives $\boldsymbol{\theta}_q - \boldsymbol{q}/h \in \mathbb{Z}^d$. Hence the only possible values of $\boldsymbol{q}$ for which $\psi \neq 0$ are determined by condition $\boldsymbol{q} \in h\mathbb{Z}^d + h\boldsymbol{\theta}_q$. The wave ψ is then necessarily a delta comb of the form

$$\psi(\boldsymbol{q}) = \sum_{\boldsymbol{s} \in h\mathbb{Z}^d + h\boldsymbol{\theta}_q} a_{\boldsymbol{s}} \delta_{\boldsymbol{s}}(\boldsymbol{q}).$$

Now

$$\psi(\boldsymbol{q} + \boldsymbol{m}_1) = \sum_{\boldsymbol{s} \in h\mathbb{Z}^d + h\boldsymbol{\theta}_q} a_{\boldsymbol{s}} \delta_{\boldsymbol{s}}(\boldsymbol{q} + \boldsymbol{m}_1) = \sum_{\boldsymbol{s} \in h\mathbb{Z}^d - \boldsymbol{m}_1 + h\boldsymbol{\theta}_q} a_{\boldsymbol{s}+\boldsymbol{m}_1} \delta_{\boldsymbol{s}}(\boldsymbol{q})$$

Thus quasiperiodicity of ψ implies that $h\mathbb{Z}^d - \boldsymbol{m}_1 = h\mathbb{Z}^d$ and $a_{\boldsymbol{s}+\boldsymbol{m}_1} = e^{2\pi i \boldsymbol{\theta}_p \cdot \boldsymbol{m}_1} a_{\boldsymbol{s}}$. The former condition implies the existence of $N \in \mathbb{Z}$ such that $h = 1/N$ and hence

$$\psi(\boldsymbol{q}) = \frac{1}{N^{d/2}} \sum_{\boldsymbol{s} \in \mathbb{Z}^d/N} c_{\boldsymbol{s}} \delta_{\boldsymbol{s} + \boldsymbol{\theta}_q/N}(\boldsymbol{q}), \tag{A.14}$$

where $c_{\boldsymbol{s}} = N^d a_{\boldsymbol{s}+\boldsymbol{\theta}_q/N}$ is a quasi-periodic sequence of arbitrary numbers supported on $\mathbb{Z}^d/N$ lattice satisfying $c_{\boldsymbol{s}+\boldsymbol{m}_1} = e^{2\pi i \boldsymbol{\theta}_p \cdot \boldsymbol{m}_1} c_{\boldsymbol{s}}$. Throughout the appendix the following normalization will be used

$$\|\psi\|_2^2 = \frac{1}{N^d} \sum_{\boldsymbol{s} \in \mathbb{Q}_N^d} |c_{\boldsymbol{s}}|^2 = 1. \tag{A.15}$$

This enables us to identify the space of all admissible quasiperiodic wave functions with the Hilbert space $\mathbb{C}^N$.

Quantum propagator

In this section, following [63] and [89], we recall the explicit formula for the planar quantum cat map propagator. We denote by $U(\boldsymbol{q}_1, \boldsymbol{q}_0)$ the kernel of the propagator i.e. the Green function for the Schrödinger equation for quantum cat map dynamics on $\mathbb{R}^2$. In order to find $U(\boldsymbol{q}_1, \boldsymbol{q}_0)$ we can either quantize the Hamiltonian (derived in Section A.1) and solve the corresponding Schrödinger equation, or use the fact that the Hamiltonian is quadratic which implies that the semiclassical approximation to the quantum propagator is exact [89] and hence the kernel can be directly expressed in terms of the classical action given by formula (A.6) in section A.1. Applying the second method we immediately get

$$\begin{aligned} U(\boldsymbol{q}_1, \boldsymbol{q}_0) &= \left(\frac{i}{2\pi\hbar}\frac{1}{b}\right)^{d/2} \exp\left(\frac{i}{\hbar}\frac{1}{2b}(a\boldsymbol{q}_0^2 - 2\boldsymbol{q}_0\boldsymbol{q}_1 + d\boldsymbol{q}_1^2)\right) \\ &= \left(\frac{Ni}{b}\right)^{d/2} \exp\left(\frac{N\pi i}{b}(a\boldsymbol{q}_0^2 - 2\boldsymbol{q}_0\boldsymbol{q}_1 + d\boldsymbol{q}_1^2)\right). \end{aligned}$$

The quantum evolution $\psi_1 = U\psi_0$ of an initial wave function ψ_0 is then given by

$$\psi_1(\boldsymbol{q}_1) = \int_{\mathbb{R}^d} U(\boldsymbol{q}_1, \boldsymbol{q}_0)\psi_0(\boldsymbol{q}_0)d\boldsymbol{q}_0.$$

In particular, for an initial wave in form of a $\boldsymbol{\theta}$-quasiperiodic delta comb

$$\psi_0(\boldsymbol{q}) = \sum_{\boldsymbol{s}\in\mathbb{Z}^d/N} c_{\boldsymbol{s}}\delta_{\boldsymbol{s}+\boldsymbol{\theta}_q/N}(\boldsymbol{q}),$$

the evolution is given by

$$\begin{aligned} \psi_1(\boldsymbol{q}_1) &= \int_{\mathbb{R}^d} U(\boldsymbol{q}_1, \boldsymbol{q}_0) \sum_{\boldsymbol{s}\in\mathbb{Z}^d/N} c_{\boldsymbol{s}}\delta_{\boldsymbol{s}+\boldsymbol{\theta}_q/N}(\boldsymbol{q}_0)d\boldsymbol{q}_0 \qquad \text{(A.16)} \\ &= \left(\frac{iN}{b}\right)^{d/2} \sum_{\boldsymbol{s}\in\mathbb{Z}^d/N} c_{\boldsymbol{s}} \exp\left(\frac{\pi i N}{b}(a(\boldsymbol{s}+\boldsymbol{\theta}_q/N)^2 - 2(\boldsymbol{s}+\boldsymbol{\theta}_q/N)\boldsymbol{q}_1 + d\boldsymbol{q}_1^2)\right). \end{aligned}$$

We will use the above formula in Appendix B where we show that the evolution of the Wigner transform of ψ_0 is given by the Frobenius-Perron operator associated with a classical cat map.

Appendix B

Wigner transform

This appendix is devoted to the Wigner transform. By means of distribution theory we derive an explicit form of its discrete version for wave functions assuming quasiperiodic delta comb form. We show that the discrete Wigner function takes the form of a uniformly spaced Dirac "delta brush", which is periodic and supported on the $2N \times 2N$ rational grid on the unit torus. We derive several important properties of the discrete Wigner transform and show that it provides a clear geometric interpretation of the quantum cat maps evolution (Proposition B.1). We conclude the appendix with a simple proof of the equivalence between canonical and algebraic finite dimensional quantizations of the cat map dynamics.

We start by recalling briefly standard definition of the Wigner transform. In order to simplify the notation we will use the symbol $\boldsymbol{x} = (\boldsymbol{q}, \boldsymbol{p})$ to denote the phase space variables and the symbol $\boldsymbol{k} = (\boldsymbol{k}_1, \boldsymbol{k}_2)$ to denote the conjugate variables in the frequency space.

Let ψ denote a wave function. The associated Wigner transform $W_\psi(\boldsymbol{q}, \boldsymbol{p})$ is given by

$$\begin{aligned} W_\psi(\boldsymbol{x}) &= W_\psi(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{h^d} \int_{\mathbb{R}^d} \psi\left(\boldsymbol{q} + \frac{\boldsymbol{q}'}{2}\right) \overline{\psi}\left(\boldsymbol{q} - \frac{\boldsymbol{q}'}{2}\right) e^{-\frac{2\pi i}{h} \boldsymbol{p}\cdot\boldsymbol{q}'} d\boldsymbol{q}' \\ &= \frac{1}{h^d} \int_{\mathbb{R}^d} \rho\left(\boldsymbol{q} + \frac{\boldsymbol{q}'}{2}, \boldsymbol{q} - \frac{\boldsymbol{q}'}{2}\right) e^{-2\pi i \frac{\boldsymbol{p}\cdot\boldsymbol{q}'}{h}} d\boldsymbol{q}', \end{aligned}$$

where $\rho(\boldsymbol{u},\boldsymbol{v}) = |\psi\rangle\langle\psi| = \psi\otimes\overline{\psi} = \psi(\boldsymbol{u})\overline{\psi}(\boldsymbol{v})$ is the density kernel associated with the wave function ψ.

In its most general form the Wigner transform is defined for any tempered distribution $\psi \in \mathcal{S}'(\mathbb{R}^n)$ or density $\rho \in \mathcal{S}'(\mathbb{R}^{2n})$. Using this fact one can derive a useful integral version of the Wigner transform. To this end, let ϕ denote an arbitrary test function from $\mathcal{S}(\mathbb{R}^d)$ and apply the change of variables

$$\boldsymbol{u} = \boldsymbol{q} + \frac{\boldsymbol{q}'}{2}, \qquad \boldsymbol{v} = \boldsymbol{q} - \frac{\boldsymbol{q}'}{2};$$

to get

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} W_\rho(\boldsymbol{x})\phi(\boldsymbol{x})d\boldsymbol{x} &= \frac{1}{h^d}\int_{\mathbb{R}^{3d}} \rho\left(\boldsymbol{q}+\frac{\boldsymbol{q}'}{2}, \boldsymbol{q}-\frac{\boldsymbol{q}'}{2}\right) e^{-\frac{2\pi i}{h}\boldsymbol{p}\cdot\boldsymbol{q}'}\phi(\boldsymbol{x})d\boldsymbol{q}'d\boldsymbol{x} \\
&= \frac{1}{h^d}\int_{\mathbb{R}^{3d}} \rho(\boldsymbol{u},\boldsymbol{v})e^{-\frac{2\pi i}{h}\boldsymbol{p}\cdot(\boldsymbol{u}-\boldsymbol{v})}\phi\left(\frac{\boldsymbol{u}+\boldsymbol{v}}{2},\boldsymbol{p}\right)d\boldsymbol{u}d\boldsymbol{v}d\boldsymbol{p} \\
&= \frac{1}{h^d}\int_{\mathbb{R}^{4d}} \rho(\boldsymbol{u},\boldsymbol{v})\delta_{\boldsymbol{q}}\left(\frac{\boldsymbol{u}+\boldsymbol{v}}{2}\right) e^{-\frac{2\pi i}{h}\boldsymbol{p}\cdot(\boldsymbol{u}-\boldsymbol{v})}\phi(\boldsymbol{x})d\boldsymbol{u}d\boldsymbol{v}d\boldsymbol{x}.
\end{aligned}$$

From above computations we see that the kernel of the Wigner transform is given by the continuous family of integral Fano operators $S_{\boldsymbol{x}}$

$$S_{\boldsymbol{x}}(\boldsymbol{u},\boldsymbol{v}) := \delta_{\boldsymbol{q}}\left(\frac{\boldsymbol{u}+\boldsymbol{v}}{2}\right) e^{\frac{2\pi i}{h}\boldsymbol{p}\cdot(\boldsymbol{u}-\boldsymbol{v})}.$$

Thus the Wigner transform $W_\rho(\boldsymbol{x})$ of ρ admits the following kernel representation

$$\begin{aligned}
W_\rho(\boldsymbol{x}) &= \frac{1}{h^d}\operatorname{Tr}\rho S_{\boldsymbol{x}} = \frac{1}{h^d}\int_{\mathbb{R}^{2d}}\rho(\boldsymbol{u},\boldsymbol{v})S_{\boldsymbol{x}}(\boldsymbol{v},\boldsymbol{u})d\boldsymbol{v}d\boldsymbol{u} \\
&= \frac{1}{h^d}\int_{\mathbb{R}^{2d}}\rho(\boldsymbol{u},\boldsymbol{v})\delta_{\boldsymbol{q}}\left(\frac{\boldsymbol{u}+\boldsymbol{v}}{2}\right) e^{-\frac{2\pi i}{h}\boldsymbol{p}\cdot(\boldsymbol{u}-\boldsymbol{v})}d\boldsymbol{u}d\boldsymbol{v}
\end{aligned}$$

Discrete Wigner function

In this section we construct a discrete version of a Wigner transform.

Using the kernel representation and applying it to a quasiperiodic distributional wave function ψ of the form (A.14) with $h = N^{-1}$ we get

$$W_\psi(\boldsymbol{x}) = \sum_{\boldsymbol{s}_1,\boldsymbol{s}_2\in\mathbb{Z}^d/N} c_{\boldsymbol{s}_1}\bar{c}_{\boldsymbol{s}_2}\delta_{\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}}(\boldsymbol{q})e^{-2\pi i N\boldsymbol{p}\cdot(\boldsymbol{s}_1-\boldsymbol{s}_2)}. \tag{B.1}$$

Our main goal is to show that (B.1) assumes a periodic "delta brush" structure. To this end we first change summation indexes in (B.1).

$$\boldsymbol{r} = \frac{\boldsymbol{s}_1 + \boldsymbol{s}_2}{2} \quad \boldsymbol{s} = \frac{\boldsymbol{s}_1 - \boldsymbol{s}_2}{2}$$

Note that the above transformation does not merely undo the change of variables performed in deriving the kernel representation of W_ψ (the Jacobian this time is non-unital).

$$\begin{aligned} W_\psi(\boldsymbol{x}) &= \sum_{\boldsymbol{r},\boldsymbol{s}\in\mathbb{Z}^d/2N} c_{\boldsymbol{r}+\boldsymbol{s}}\bar{c}_{\boldsymbol{r}-\boldsymbol{s}}\delta_{\boldsymbol{r}+\frac{\boldsymbol{\theta}_q}{N}}(\boldsymbol{q})e^{-2\pi i2N\boldsymbol{ps}} \\ &= \sum_{\boldsymbol{r}\in\mathbb{Z}^d/2N} \delta_{\boldsymbol{r}+\frac{\boldsymbol{\theta}_q}{N}}(\boldsymbol{q}) \sum_{\boldsymbol{s}\in\mathbb{Z}^d/2N} c_{\boldsymbol{r}+\boldsymbol{s}}\bar{c}_{\boldsymbol{r}-\boldsymbol{s}}e^{-2\pi i2N\boldsymbol{ps}} \end{aligned}$$

(We use a convention that $c_q = 0$ whenever $q \notin \mathbb{Z}/N$.) Next we have

$$\begin{aligned} \sum_{\boldsymbol{s}\in\mathbb{Z}^d/2N} c_{\boldsymbol{r}+\boldsymbol{s}}\bar{c}_{\boldsymbol{r}-\boldsymbol{s}}e^{-2\pi i2N\boldsymbol{ps}} &= \sum_{\boldsymbol{t}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N\boldsymbol{pt}} \sum_{\boldsymbol{s}\in\mathbb{Z}^d} e^{-2\pi i2N(\boldsymbol{p}-\frac{\boldsymbol{\theta}_p}{N})\cdot\boldsymbol{s}} \\ = \sum_{\boldsymbol{t}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N\boldsymbol{pt}} &\sum_{\boldsymbol{s}\in\mathbb{Z}^d} \delta_{\boldsymbol{s}}(2N(\boldsymbol{p}-\frac{\boldsymbol{\theta}_p}{N})) \\ = \sum_{\boldsymbol{t}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N\boldsymbol{pt}} &\frac{1}{(2N)^d} \sum_{\boldsymbol{s}\in\mathbb{Z}^d/2N} \delta_{\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N}}(\boldsymbol{p}) \end{aligned}$$

Thus

$$\begin{aligned} W_\psi(\boldsymbol{x}) &= \frac{1}{(2N)^d} \sum_{\boldsymbol{r},\boldsymbol{s}\in\mathbb{Z}^d/2N} \sum_{\boldsymbol{t}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\cdot\boldsymbol{t}}\delta_{\boldsymbol{r}+\frac{\boldsymbol{\theta}_q}{N}}(\boldsymbol{q})\delta_{\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N}}(\boldsymbol{p}) \\ &= \frac{1}{(2N)^d} \sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\cdot\boldsymbol{t}}\tilde{\delta}_{\boldsymbol{r}+\frac{\boldsymbol{\theta}_q}{N}}(\boldsymbol{q})\tilde{\delta}_{\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N}}(\boldsymbol{p}). \end{aligned}$$

The above formulas provide a clear geometric interpretation of the Wigner transform of a quasiperiodic delta comb. A few remarks are in order here. First of all, the resulting Wigner function is strictly periodic (even though the original wave need not be) and is supported on the grid with a mesh spacing of the size of $h/2$ (half of the corresponding resolution for the wave). The support of the Wigner transform forms a

$2N\times 2N$ lattice centered at (shifted from the origin by) $\boldsymbol{\theta}$. The later property explains the interpretation of $\boldsymbol{\theta}$ as Bloch or Floquet "angles" (cf. Section 5.1.4) and the term "quantum boundary conditions" coined in [70] in the context of the quantization condition 5.20 (see Proposition 5.10 in Section 5.2.1).

Now we want to find discrete Fourier coefficients of W_ψ. Taking the Fourier transform in this case requires some care, since one has to adjust the value of the Planck constant adequately. Taking into account the support of W_ψ, its discrete Fourier transform agrees with a distributional one if the latter is taken with the value of the Planck constant equal to $h/2$. Indeed, for any $\boldsymbol{k}\in\mathbb{Z}^{2d}/2N$, we have

$$\hat{W}_\psi(\boldsymbol{k}) = \frac{1}{(2N)^{2d}} \sum_{\boldsymbol{r},\boldsymbol{s},\boldsymbol{t}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}} e^{-2\pi i 2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\boldsymbol{t}} e^{-2\pi i 2N\boldsymbol{k}_1\cdot(\boldsymbol{r}+\frac{\boldsymbol{\theta}_q}{N})} e^{-2\pi i 2N\boldsymbol{k}_2\cdot(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})}. \tag{B.2}$$

The above formula can be simplified as follows

$$\begin{aligned}
\hat{W}_\psi(\boldsymbol{k}) &= \frac{1}{(2N)^{2d}} \sum_{\boldsymbol{r},\boldsymbol{t}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}} e^{-2\pi i 2N\boldsymbol{k}_1\cdot(\boldsymbol{r}+\frac{\boldsymbol{\theta}_q}{N})} \sum_{\boldsymbol{s}\in\mathbb{Q}^d_{2N}} e^{-2\pi i 2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\cdot(\boldsymbol{t}+\boldsymbol{k}_2)} \\
&= \frac{1}{(2N)^{d}} \sum_{\boldsymbol{r}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}-\boldsymbol{k}_2}\bar{c}_{\boldsymbol{r}+\boldsymbol{k}_2} e^{-2\pi i 2N\boldsymbol{k}_1\cdot(\boldsymbol{r}+\frac{\boldsymbol{\theta}_q}{N})}.
\end{aligned}$$

For further simplification one needs to note that half of the coefficients are zero, since $c_{\boldsymbol{r}}$ are supported on $\mathbb{Z}/N$. Moreover due to the quasiperiodicity of $c_{\boldsymbol{r}}$ the product $c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}}$ is periodic. We can thus apply the change of indices $\boldsymbol{r}=\boldsymbol{t}-\boldsymbol{k}_2$ to obtain

$$\begin{aligned}
\hat{W}_\psi(\boldsymbol{k}) &= \frac{1}{(2N)^{d}} \sum_{\boldsymbol{r}\in\mathbb{Q}^d_{2N}-\boldsymbol{k}_2} c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}+2\boldsymbol{k}_2} e^{-2\pi i 2N\boldsymbol{k}_1(\boldsymbol{r}+\boldsymbol{k}_2+\frac{\boldsymbol{\theta}_q}{N})} \\
&= \frac{1}{(2N)^{d}} \sum_{\boldsymbol{r}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}+2\boldsymbol{k}_2} e^{-2\pi i 2N\boldsymbol{k}_1(\boldsymbol{r}+\boldsymbol{k}_2+\frac{\boldsymbol{\theta}_q}{N})} \\
&= \frac{1}{(2N)^{d}} e^{-2\pi i 2N\boldsymbol{k}_1(\boldsymbol{k}_2+\frac{\boldsymbol{\theta}_q}{N})} \sum_{\boldsymbol{r}\in\mathbb{Q}^d_{N}} c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}+2\boldsymbol{k}_2} e^{-2\pi i N\boldsymbol{k}_1\cdot\boldsymbol{r}}.
\end{aligned}$$

Thus we conclude that the discrete Fourier coefficients of W_ψ are given by the so-called

discrete ambiguity function.

$$\hat{W}_\psi(\boldsymbol{k}) = \frac{1}{(2N)^d}\sum_{\boldsymbol{r}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}-\boldsymbol{k}_2}\bar{c}_{\boldsymbol{r}+\boldsymbol{k}_2} e^{-2\pi i 2N\boldsymbol{k}_1\cdot(\boldsymbol{r}+\frac{\boldsymbol{\theta}_q}{N})} \tag{B.3}$$

$$= \frac{1}{(2N)^d}\sum_{\boldsymbol{r}\in\mathbb{Q}^d_{N}} c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}+2\boldsymbol{k}_2} e^{-2\pi i 2N\boldsymbol{k}_1(\boldsymbol{r}+\boldsymbol{k}_2+\frac{\boldsymbol{\theta}_q}{N})}. \tag{B.4}$$

From the above formula we see that $\hat{W}(\boldsymbol{k})$ is 1-quasiperiodic in both variables and is supported on $\mathbb{Z}^d/2N$ lattice.

Properties of the Discrete Wigner function

It is well known that in the continuous setting the Wigner function can be viewed as a quantum counterpart of the joint phase space density for the position and momentum variables. For any $\psi \in L^2(\mathbb{R}^d)$ satisfying normalization condition $\|\psi\|_2 = 1$ one has

$$\int_{\mathbb{R}^d} W_\psi(\boldsymbol{x})d\boldsymbol{p} = |\psi(\boldsymbol{q})|^2, \quad \int_{\mathbb{R}^d} W_\psi(\boldsymbol{x})d\boldsymbol{q} = |\mathcal{F}_h\psi(\boldsymbol{p})|^2, \quad \int_{\mathbb{R}^{2d}} W_\psi(\boldsymbol{x})d\boldsymbol{x} = 1,$$

The Wigner function is always real but need not be nonnegative (one can easily note that $W_\psi(\boldsymbol{0}) = -\left(\frac{2}{h}\right)^d \|\psi(\boldsymbol{q})\|_2^2$ is negative for any odd ψ). However, the following useful property (Parseval identity) holds

$$0 \le \langle W_{\psi_1}, W_{\psi_2}\rangle_{L^2(\mathbb{R}^{2d})} = \frac{1}{h^d}|\langle \psi_1, \psi_2\rangle_{L^2(\mathbb{R}^d)}|^2 \le \frac{1}{h^d}.$$

Not all of these properties can be generalized in any obvious way to the whole $\mathcal{S}'(\mathbb{R})$ since the waves from outside of $L^2(\mathbb{R})$ are no longer normalizable. Nevertheless, as we show in this section, all these properties are preserved for quasiperiodic waves. There are also some interesting characteristic properties of discrete Wigner function (not shared by the standard continuous version). Below we mention two of them

I. $W(\boldsymbol{x})$ is completely determined by $(2N)^{2d}$ matrix of the strengths of its delta functions.

$$w_{\boldsymbol{r},\boldsymbol{s}} = \frac{1}{(2N)^d}\sum_{\boldsymbol{t}\in\mathbb{Q}^d_{2N}} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}} e^{-2\pi i 2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\cdot\boldsymbol{t}}.$$

II. In general all $(2N)^{2d}$ coefficients may be nonzero, but at the same time the coefficients attached to a vertices of a $\frac{1}{2}$-box may differ at most in sign.

$$\begin{aligned} w_{\boldsymbol{r}+\frac{\boldsymbol{e}_i}{2},\boldsymbol{s}} &= \frac{1}{(2N)^d}\sum_{\boldsymbol{t}\in\mathbb{Q}_{2N}^d} c_{\boldsymbol{r}+\frac{\boldsymbol{e}_i}{2}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}+\frac{\boldsymbol{e}_i}{2}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\cdot\boldsymbol{t}} \\ &= \frac{1}{(2N)^d}\sum_{\boldsymbol{t}\in\mathbb{Q}_{2N}^d} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{\theta}_{p_i}/2N)}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\cdot(\boldsymbol{t}-\frac{\boldsymbol{e}_i}{2})} \\ &= e^{\pi i2N\boldsymbol{s}\cdot\boldsymbol{e}_i}\frac{1}{(2N)^d}\sum_{\boldsymbol{t}\in\mathbb{Q}_{2N}^d} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\cdot\boldsymbol{t}} = (-1)^{2N\boldsymbol{s}\cdot\boldsymbol{e}_i}w_{\boldsymbol{r},\boldsymbol{s}} \end{aligned}$$

and

$$\begin{aligned} w_{\boldsymbol{r},\boldsymbol{s}+\frac{\boldsymbol{e}_i}{2}} &= \frac{1}{(2N)^d}\sum_{\boldsymbol{t}\in\mathbb{Q}_{2N}^d} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N}+\frac{\boldsymbol{e}_i}{2})\boldsymbol{t}} \\ &= \frac{1}{(2N)^d}\sum_{\boldsymbol{t}\in\mathbb{Q}_{2N}^d} e^{-\pi i2N\boldsymbol{t}\cdot\boldsymbol{e}_i}c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\boldsymbol{t}} \\ &= \frac{1}{(2N)^d}\sum_{\boldsymbol{t}\in\mathbb{Q}_{2N}^d} (-1)^{2N\boldsymbol{t}\cdot\boldsymbol{e}_i}c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\boldsymbol{t}}. \end{aligned}$$

Now it suffices to note that $c_{\boldsymbol{r}+\boldsymbol{t}}$ are zero whenever $\boldsymbol{r}\cdot\boldsymbol{e}_i$ and $\boldsymbol{t}\cdot\boldsymbol{e}_i$ are of different parity which gives

$$w_{\boldsymbol{r},\boldsymbol{s}+\frac{1}{2}} = (-1)^{2N\boldsymbol{r}\cdot\boldsymbol{e}_i}\frac{1}{(2N)^d}\sum_{\boldsymbol{t}\in\mathbb{Q}_{2N}^d} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\boldsymbol{t}} = (-1)^{2N\boldsymbol{r}\cdot\boldsymbol{e}_i}w_{\boldsymbol{r},\boldsymbol{s}}.$$

Next we derive the properties regarding marginal distributions

III. $\boldsymbol{q}$-marginal projection.

$$\begin{aligned} \sum_{\boldsymbol{s}\in\mathbb{Q}_{2N}^d} w_{\boldsymbol{r},\boldsymbol{s}} &= \frac{1}{(2N)^d}\sum_{\boldsymbol{s}\in\mathbb{Q}_{2N}^d}\sum_{\boldsymbol{t}\in\mathbb{Q}_{2N}^d} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\boldsymbol{t}} \\ &= \frac{1}{(2N)^d}\sum_{\boldsymbol{t}\in\mathbb{Q}_{2N}^d} c_{\boldsymbol{r}+\boldsymbol{t}}\bar{c}_{\boldsymbol{r}-\boldsymbol{t}}\sum_{\boldsymbol{s}\in\mathbb{Q}_{2N}^d} e^{-2\pi i2N(\boldsymbol{s}+\frac{\boldsymbol{\theta}_p}{N})\boldsymbol{t}} = c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}} = |c_{\boldsymbol{r}}|^2 \end{aligned}$$

IV. $\boldsymbol{p}$-marginal projection results in $|\hat{c}_{\boldsymbol{s}}|^2$ (we omit the proof, which is in this case a bit more technically involved).

We end this section by deriving the discrete version of the Parseval identity for the Wigner function. That is, we want to prove the following identity

V.

$$\sum_{\boldsymbol{k}\in\mathbb{Q}_{2N}^{2d}} \hat{W}_{\psi_1}(\boldsymbol{k})\hat{W}_{\psi_2}(\boldsymbol{k}) = N^d \left| \frac{1}{N^d} \sum_{\boldsymbol{r}\in\mathbb{Q}_N^d} \bar{c}_{\boldsymbol{r}} d_{\boldsymbol{r}} \right|^2 .$$

In the prove we use the Fourier transform of W_ψ derived in the previous section.

$$\begin{aligned}
&\sum_{\boldsymbol{k}\in\mathbb{Q}_{2N}^{2d}} \hat{W}_{\psi_1}(\boldsymbol{k})\hat{W}_{\psi_2}(\boldsymbol{k}) \\
= \; & \frac{1}{(2N)^{2d}} \sum_{\boldsymbol{k}\in\mathbb{Q}_{2N}^{2d}} \sum_{\boldsymbol{r},\boldsymbol{s}\in\mathbb{Q}_N^d} c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}+2\boldsymbol{k}_2} e^{-2\pi i 2N\boldsymbol{k}_1\cdot(\boldsymbol{r}+\boldsymbol{k}_2+\frac{\boldsymbol{\theta}_q}{N})} \bar{d}_{\boldsymbol{s}} d_{\boldsymbol{s}+2\boldsymbol{k}_2} e^{2\pi i 2N\boldsymbol{k}_1\cdot(\boldsymbol{s}+\boldsymbol{k}_2+\frac{\boldsymbol{\theta}_q}{N})} \\
= \; & \frac{1}{(N)^{2d}} \sum_{\boldsymbol{k}\in\mathbb{Q}_N^{2d}} \sum_{\boldsymbol{r},\boldsymbol{s}\in\mathbb{Q}_N^d} c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}+\boldsymbol{k}_2} e^{-2\pi i N\boldsymbol{k}_1\cdot\boldsymbol{r}} \bar{d}_{\boldsymbol{s}} d_{\boldsymbol{s}+\boldsymbol{k}_2} e^{2\pi i N\boldsymbol{k}_1\cdot\boldsymbol{s}} \\
= \; & \frac{1}{N^{2d}} \sum_{\boldsymbol{k}_2,\boldsymbol{r},\boldsymbol{s}\in\mathbb{Q}_N^d} c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}+\boldsymbol{k}_2} \bar{d}_{\boldsymbol{s}} d_{\boldsymbol{s}+\boldsymbol{k}_2} \sum_{\boldsymbol{k}_1\in\mathbb{Q}_N^d} e^{-2\pi i N\boldsymbol{k}_1\cdot(\boldsymbol{r}-\boldsymbol{s})} \\
= \; & \frac{1}{N^{2d}} \sum_{\boldsymbol{k}_2,\boldsymbol{r},\boldsymbol{s}\in\mathbb{Q}_N^d} c_{\boldsymbol{r}}\bar{c}_{\boldsymbol{r}+\boldsymbol{k}_2} \bar{d}_{\boldsymbol{s}} d_{\boldsymbol{s}+\boldsymbol{k}_2} N^d \delta(\boldsymbol{r}-\boldsymbol{s}) = N^d \left| \frac{1}{N^d} \sum_{\boldsymbol{r}\in\mathbb{Q}_N^d} \bar{c}_{\boldsymbol{r}} d_{\boldsymbol{r}} \right|^2 .
\end{aligned}$$

Cat map evolution of the Wigner function

In this section we show the the Wigner function associated with a quasiperiodic delta comb evolves classically under the cat map dynamics. Indeed, we have the following

Proposition B.1 *Let U denote the quantum propagator associated with a cat map F. For any initial $\boldsymbol{\theta}$-quasiperiodic delta wave ψ_0, the Wigner transform of the evolved wave $\psi_1 = U\psi_0$ satisfies the property $W_{\psi_1} = W_{\psi_0} \circ F^{-1}$.*

Proof. The prove is obtained by a direct application of formula (A.16) (see Section A.2) which gives

$$\begin{aligned}
W_{\psi_1}(\boldsymbol{x}) = \int_{\mathbb{R}^d} \left(\frac{Ni}{b}\right)^{d/2} \sum_{\boldsymbol{s}_1} c_{\boldsymbol{s}_1} e^{\frac{N\pi i}{b}\left(a(\boldsymbol{s}_1+\frac{\boldsymbol{\theta}_q}{N})^2 - 2(\boldsymbol{s}_1+\frac{\boldsymbol{\theta}_q}{N})\left(\boldsymbol{q}+\frac{\boldsymbol{q}'}{2}\right) + d\left(\boldsymbol{q}+\frac{\boldsymbol{q}'}{2}\right)^2\right)} \\
\left(-\frac{Ni}{b}\right)^{d/2} \sum_{\boldsymbol{s}_2} \bar{c}_{\boldsymbol{s}_2} e^{-\frac{N\pi i}{b}\left(a(\boldsymbol{s}_2+\frac{\boldsymbol{\theta}_q}{N})^2 - 2(\boldsymbol{s}_2+\frac{\boldsymbol{\theta}_q}{N})\left(\boldsymbol{q}-\frac{\boldsymbol{q}'}{2}\right) + d\left(\boldsymbol{q}-\frac{\boldsymbol{q}'}{2}\right)^2\right)} e^{-2\pi i N\boldsymbol{p}\boldsymbol{q}'} d\boldsymbol{q}'
\end{aligned}$$

From the above we can represent $W_{\psi_1}(\boldsymbol{x})$ by

$$\frac{N^d}{b}\sum_{\boldsymbol{s}_1,\boldsymbol{s}_2} c_{\boldsymbol{s}_1}\bar{c}_{\boldsymbol{s}_2}\int_{\mathbb{R}^d} e^{\frac{2\pi iN}{b}\left(a\frac{\boldsymbol{s}_1^2-\boldsymbol{s}_2^2}{2}+a(\boldsymbol{s}_1-\boldsymbol{s}_2)\frac{\boldsymbol{\theta}_q}{N}-(\boldsymbol{s}_1-\boldsymbol{s}_2)\boldsymbol{q}-\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}\boldsymbol{q}'-\boldsymbol{q}'\frac{\boldsymbol{\theta}_q}{N}+d\boldsymbol{q}\boldsymbol{q}'-b\boldsymbol{p}\boldsymbol{q}'\right)}d\boldsymbol{q}'$$
$$= \sum_{\boldsymbol{s}_1,\boldsymbol{s}_2} c_{\boldsymbol{s}_1}\bar{c}_{\boldsymbol{s}_2}\int_{\mathbb{R}^d} e^{2\pi i\boldsymbol{q}'\left(d\boldsymbol{q}-b\boldsymbol{p}-\left(\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}\right)\right)}d\boldsymbol{q}' e^{\frac{2\pi iN}{b}\left(a\frac{\boldsymbol{s}_1^2-\boldsymbol{s}_2^2}{2}+a(\boldsymbol{s}_1-\boldsymbol{s}_2)\frac{\boldsymbol{\theta}_q}{N}-(\boldsymbol{s}_1-\boldsymbol{s}_2)\boldsymbol{q}\right)}.$$

Thus using the spectral resolution of the Dirac delta comb we get

$$W_{\psi_1}(\boldsymbol{x}) = \sum_{\boldsymbol{s}_1,\boldsymbol{s}_2} c_{\boldsymbol{s}_1}\bar{c}_{\boldsymbol{s}_2}\delta_{\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}}(d\boldsymbol{q}-b\boldsymbol{p})e^{\frac{2\pi iN}{b}\left(a\left(\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}\right)-\boldsymbol{q}\right)(\boldsymbol{s}_1-\boldsymbol{s}_2)}$$
$$= \sum_{\boldsymbol{s}_1,\boldsymbol{s}_2} c_{\boldsymbol{s}_1}\bar{c}_{\boldsymbol{s}_2}\delta_{\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}}(d\boldsymbol{q}-b\boldsymbol{p})e^{2\pi iN\left(\frac{a}{b}\left(\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}\right)-\frac{ad-bc}{b}\boldsymbol{q}\right)(\boldsymbol{s}_1-\boldsymbol{s}_2)}$$
$$= \sum_{\boldsymbol{s}_1,\boldsymbol{s}_2} c_{\boldsymbol{s}_1}\bar{c}_{\boldsymbol{s}_2}\delta_{\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}}(d\boldsymbol{q}-b\boldsymbol{p})e^{2\pi iN\left(\frac{a}{b}\left(\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}\right)-\frac{a}{b}\left(\left(\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}\right)+b\boldsymbol{p}+c\boldsymbol{q}\right)(\boldsymbol{s}_1-\boldsymbol{s}_2)}$$
$$= \sum_{\boldsymbol{s}_1,\boldsymbol{s}_2} c_{\boldsymbol{s}_1}\bar{c}_{\boldsymbol{s}_2}\delta_{\frac{\boldsymbol{s}_1+\boldsymbol{s}_2}{2}+\frac{\boldsymbol{\theta}_q}{N}}(d\boldsymbol{q}-b\boldsymbol{p})e^{2\pi iN(-c\boldsymbol{q}+a\boldsymbol{p})(\boldsymbol{s}_1-\boldsymbol{s}_2)} = W_{\psi_0}\circ F^{-1}(\boldsymbol{x})$$

Wigner function, Weyl quantization and ambiguity function

We end this appendix by recalling briefly the relation between the Wigner function, the Weyl quantization and the ambiguity function. We have

Proposition B.2 *For any classical observable $f \in L^2(\mathbb{R}^{2d})$ and any wave function $\psi \in L^2(\mathbb{R}^{2d})$, the expectation of the Weyl quantization $Op(f)$ of f satisfies*

$$\langle\psi, Op(f)\psi\rangle = \int_{R^{2d}} f(\boldsymbol{x})W_\psi(\boldsymbol{x})d\boldsymbol{x}.$$

In particular, the expectation of any Weyl translation operator $T_{\boldsymbol{k}}$ (cf. (5.2)*, Section 5.1.1) is given by the ambiguity function $A_\psi(\boldsymbol{k})$ (the inverse Fourier transform of the Wigner function) associated with a wave ψ*

$$\langle\psi, T_{\boldsymbol{k}}\psi\rangle = A_\psi(\boldsymbol{k}) = \int_{R^{2d}} W_\psi(\boldsymbol{x})e^{\frac{2\pi i}{h}\boldsymbol{k}\wedge\boldsymbol{x}}d\boldsymbol{x}.$$

Proof. Weyl translations generate the whole algebra of observables. It is thus enough to prove the second statement. Using the formula (5.3) for the explicit action of Weyl

translations on wave functions derived in section 5.1.1 we get

$$\begin{aligned}\langle\psi, T_{\boldsymbol{k}}\psi\rangle &= \int_{R^d}\overline{\psi}(\boldsymbol{q})T_k\psi(\boldsymbol{q})d\boldsymbol{q} = \int_{R^d}\overline{\psi}(\boldsymbol{q})e^{\frac{2\pi i}{h}\boldsymbol{k}_2\cdot(\boldsymbol{q}-\boldsymbol{k}_1/2)}\psi(\boldsymbol{q}-\boldsymbol{k}_1)d\boldsymbol{q} \\ &= \int_{R^d}\psi(\boldsymbol{q}-\boldsymbol{k}_1/2)\overline{\psi}(\boldsymbol{q}+\boldsymbol{k}_1/2))e^{\frac{2\pi i}{h}\boldsymbol{k}_2\cdot\boldsymbol{q}}d\boldsymbol{q} = A_\psi(\boldsymbol{k})\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{R^{2d}}W_\psi(\boldsymbol{x})e^{2\pi i\frac{\boldsymbol{k}\wedge\boldsymbol{x}}{h}}d\boldsymbol{x} &= \frac{1}{h^d}\int_{R^{3d}}\psi(\boldsymbol{q}+\boldsymbol{q}'/2)\overline{\psi}(\boldsymbol{q}-\boldsymbol{q}'/2))e^{-\frac{2\pi i}{h}(\boldsymbol{p}\cdot\boldsymbol{q}'-\boldsymbol{k}_2\cdot\boldsymbol{q}+\boldsymbol{k}_1\cdot\boldsymbol{p})}d\boldsymbol{q}'d\boldsymbol{x} \\ &= \int_{R^d}\psi(\boldsymbol{q}-\boldsymbol{k}_1/2)\overline{\psi}(\boldsymbol{q}+\boldsymbol{k}_1/2))e^{\frac{i}{h}\boldsymbol{k}_2\cdot\boldsymbol{q}}d\boldsymbol{q} = A_\psi(\boldsymbol{k}),\end{aligned}$$

which completes the proof. ■

Remark B.3 *The above proposition generalizes to the wave functions from $\mathcal{S}'(\mathbb{R}^d)$ (in this case the observables need to be taken from a smaller set). In particular, the Proposition holds for quasiperiodic delta combs and the observables from $L^2(\mathbb{T}^{2d})$, satisfying $\sum_{\boldsymbol{k}}|\hat{f}(\boldsymbol{k})| < \infty$.*

The above proposition together with Proposition B.1 provides the proof that the algebraic quantization introduced in Section 5.2.1 coincides with a canonical one, derived in Section A.2 of Appendix A. Indeed, denoting by U the propagator obtained by means of the canonical quantization and by $\mathcal{U}$ the *-automorphism constructed in the algebraic approach, we get (using symplecticity of F) what follows

$$\begin{aligned}\langle\psi_0,\mathcal{U}T_{\boldsymbol{k}}\psi_0\rangle &= \langle\psi_0, T_{F^{-1}\boldsymbol{k}}\psi_0\rangle = \int_{R^{2d}}W_{\psi_0}(\boldsymbol{x})e^{2\pi i\frac{F^{-1}\boldsymbol{k}\wedge\boldsymbol{x}}{h}}d\boldsymbol{x} \\ &= \int_{R^{2d}}W_{\psi_0}(F^{-1}\boldsymbol{x})e^{2\pi i\frac{\boldsymbol{k}\wedge\boldsymbol{x}}{h}}d\boldsymbol{x} = \int_{R^{2d}}W_{\psi_1}(\boldsymbol{x})e^{2\pi i\frac{\boldsymbol{k}\wedge\boldsymbol{x}}{h}}d\boldsymbol{x} = \langle\psi_1, T_{\boldsymbol{k}}\psi_1\rangle \\ &= \langle U\psi_0, T_{\boldsymbol{k}}U\psi_0\rangle = \langle\psi_0, U^*T_{\boldsymbol{k}}U\psi_0\rangle.\end{aligned}$$

Thus the canonical quantum propagator U implements the algebraic *-automorphism $\mathcal{U}$ on the space of quasiperiodic delta waves.

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